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On a Generalisation of the Laplace Transformation

By A. Erdélyi

(*Received* 13*th June* 1950. *Read* 3*rd November* 1950.)

1. On reading a recent paper by R. S. Varma (Varma 1949) I recalled that in May 1942 I investigated an integral transformation which is very similar to Varma's. Varma has

$$\phi_m^k(st) = \int_0^\infty (2st)^{-\frac{1}{4}} W_{k,m}(2st) f(t)\, dt \tag{1}$$

and points out that this reduces to a Laplace integral for $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$. Instead of (1), one could consider the integral

$$\int_0^\infty e^{-\frac{1}{2}st} W_{k,m}(st)\ (st)^{-k} f(t)\, dt \tag{2}$$

which was introduced by C. S. Meijer (Meijer 1940 b); this integral reduces to a Laplace integral whenever $k = m + \frac{1}{2}$. Now, apart from comparatively unimportant factors, the nucleus of (2) is a fractional derivative or integral, as the case may be, of e^{-st}, and on carrying out a fractional integration by parts, it appears that (2) is essentially the Laplace transform of a fractional integral or derivative of f. Thus, the whole theory of the transformation (2), including inversion formulae, representation theorems, etc., can be deduced from the well-known theory of the Laplace transformation. It is not quite clear that a similar reduction is possible for (1), although it is certainly possible when $k = 0$.

My work of 1942 remained unpublished, and I still hope to describe it in more detail on some future occasion. Meanwhile, in view of the reviving interest in the subject,[1] I should like to establish briefly the connection between (2) and the Laplace transformation.

2. I shall use the operators of fractional integration and differentiation whose theory has been developed by H. Kober and myself. For the sake of brevity, I shall formulate all results for the class $L_2(0, \infty)$ and merely remark that corresponding results are known for the classes L_p, $1 \leqq p \leqq \infty$. The definition of the operators in the simplest case is (Kober 1940)

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[1] Cf. for instance, a series of papers by S. K. Bose, a pupil of Dr Varma's.

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[1] Cf. for instance, a series of papers by S. K. Bose, a pupil of Dr Varma's.

$$I^{+}_{\eta,\alpha} f(x) = \{\Gamma(\alpha)\}^{-1} x^{-\eta-\alpha} \int_0^x (x-u)^{\alpha-1} u^{\eta} f(u)\, du \tag{3}$$

$$K^{-}_{\eta,\alpha} f(x) = \{\Gamma(\alpha)\}^{-1} x^{\eta} \int_x^{\infty} (u-x)^{\alpha-1} u^{-\eta-\alpha} f(u)\, du \tag{4}$$

where $f(x)$ is in $L_2(0, \infty)$, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\eta > -\frac{1}{2}$.

First, the definitions can be extended to other values of η, as long as $\operatorname{Re}\eta - \frac{1}{2}$ is not a negative integer (Erdélyi 1940); next follows the extension to $\operatorname{Re}\alpha = 0$ which is much more difficult (Kober 1941). The domain of these extensions is still the full class L_2. Lastly, the extension to $\operatorname{Re}\alpha < 0$ is given by the definition

$$I^{+}_{\eta,\alpha} = (I^{+}_{\eta+\alpha,-\alpha})^{-1}, \qquad K_{\eta,\alpha} = (K_{\eta+\alpha,-\alpha})^{-1}.$$

In this last extension it is necessary to contract the domain of definition from the full L_2 to a class $L_2^{(\alpha)}$ which coincides with L^2 if $\operatorname{Re}\alpha \geqq 0$. Here the operators will be used in the extended sense (Erdélyi 1940).

We define the Mellin transform as

$$\mathfrak{M}_t f(x) = \text{l. i. m.}_{X\to\infty} \int_{X^{-1}}^{X} x^{-\frac{1}{2}+it} f(x)\, dx,$$

where the right-hand side is a limit in mean square. For the extended operators we then have

$$\mathfrak{M}_t I^{+}_{\eta,\alpha} f = \frac{\Gamma(\eta+\frac{1}{2}-it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)} \mathfrak{M}_t f \tag{5}$$

$$\mathfrak{M}_t K^{-}_{\eta,\alpha} f = \frac{\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} \mathfrak{M}_t f. \tag{6}$$

Moreover, we have the formula for fractional integration by parts

$$\int_0^{\infty} dx\, \phi(x)\, I^{+}_{\eta,\alpha} f(x) = \int_0^{\infty} dx\, f(x)\, K^{-}_{\eta,\alpha} \phi(x), \tag{7}$$

valid if both $f(x)$ and $\phi(x)$ belong to $L_2^{(\alpha)}$.

3. After these preliminaries we define the nucleus of our transform

$$k(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\eta+\frac{1}{2}+it)\,\Gamma(\rho+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} z^{-\frac{1}{2}-it}\, dt \tag{8}$$

where we assume that neither $\operatorname{Re}\eta - \frac{1}{2}$ nor $\operatorname{Re}\rho - \frac{1}{2}$ is a negative integer. The evaluation of (8) as the sum of residues leads to an expression in terms of confluent hypergeometric functions or "cut" confluent hypergeometric functions; and from Mellin's inversion formula we have

$$\mathfrak{M}_t k(z) = \frac{\Gamma(\eta+\frac{1}{2}+it)\,\Gamma(\rho+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} = \frac{\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} \mathfrak{M}_t(z^{\rho} e^{-z}),$$

and thus $$k(z) = K^{-}_{\eta,\alpha}(z^{\rho} e^{-z}). \tag{9}$$

We can now integrate by parts according to (7) and find on account of (9) that

$$g(x) = \int_0^{\infty} k(xy) f(y)\, dy = \int_0^{\infty} e^{-xy}(xy)^{\rho} I^{+}_{\eta,\alpha} f(y)\, dy \tag{10}$$

and hence the reduction of the k-transform of f (in $L_2^{(a)}$) to x^{ρ} times the Laplace transform of $y^{\rho} I^{+}_{\eta,\alpha} f(y)$. It is also possible to prove (although this requires a justification, by absolute convergence, of the interchange of the order of integrations) that for the function defined by (10)

$$K^{-}_{\eta+\alpha,-\alpha} g(x) = \int_0^{\infty} e^{-xy}(xy)^{\rho} f(y)\, dy \tag{11}$$

for all functions f in L_2. This latter form enables one to invert the k-transformation by means of any of the numerous inversion formulae of the Laplace transformation. For representation theorems, (10) is the more suitable form.

In my unpublished work, I developed the theory of the more general transformation whose nucleus is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma\{(\eta+\frac{1}{2}+it)/n\}\,\Gamma(\rho+\frac{1}{2}+it)}{\Gamma\{\alpha+(\eta+\frac{1}{2}+it)/n\}} (xy)^{-\frac{1}{2}-it}\, dt$$

where n is any positive number, not necessarily integer. $n = 1$ is the nucleus (8), $n = 2$ leads to the particular case $k = 0$ of (1): this particular case has been studied in some detail (Meijer 1940a, Boas 1942a, b).

[*Added 20th September* 1954. Since this note was submitted for publication, a further instalment has appeared in *Rend. Sem. Mat. Università e Politecnico di Torino* 10 (1950/51), 217-234 A transformation which is equivalent to (2) has also been investigated by K. P. Bhatnagar, *Ganita* 3 (1952), 13-18, who refers to unpublished work by R. S. Varma.]

REFERENCES.

Boas, R. P., 1942a. *Proc. Nat. Ac. Sci.* 28, 21-24.
1942b. *Bull. American Math. Soc.* 48, 286-294.
Erdélyi, A., 1940. *Quart. J. Math.* (Oxford) 11, 293-303
Kober, H., 1940. *Quart. J. Math.* (Oxford) 11, 193-211.
1941. *Trans. American Math. Soc.* 50, 160-174.
Meijer, C. S., 1940a. *Proc. Amsterdam Ak. Wet.* 43, 599-608 and 702-711.
1940b. *Proc. Amsterdam Ak. Wet.* 44, 727-737 and 831-839.
Varma, R. S., 1949. *Proc. Edinburgh Math. Soc.* (2) 8, 126-127.

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On Equivalent Truth-Tables of Many-Valued Logics

By J. KALICKI

(*Received 22nd June* 1950. *Read 1st December* 1950.)

Many-valued or non-Aristotelian calculi of propositions (logics) were originally introduced by generalisation of the truth-table method. It was known by the end of the nineteenth century that ordinary "binary" formulae of the calculus of propositions, such as

$$p \to p \tag{1}$$

$$(p \to q) \to ((q \to r) \to (p \to r)) \tag{2}$$

could be verified directly by means of the truth-table:

$$A_2 \equiv \begin{array}{c|cc} p \to q & 1 & 2 \\ \hline *1 & 1 & 2 \\ 2 & 1 & 1 \\ p & & \end{array} \tag{3}$$

although the terminology and symbolism used were different.[1]

To decide, for example, whether (2) is a formula of binary logic, the numbers 1 and 2 are substituted in all possible ways for the "variables" p, q, r of (2) and the resulting expressions are contracted by writing

$$\text{number 2 for } 1 \to 2$$

$$\text{and number 1 for } 1 \to 1,\ 2 \to 1, \text{ and } 2 \to 2$$

in accordance with table (3). If (2) contracts to 1 for all possible substitutions, then (2) is a "theorem" of the binary logic of implication; otherwise it is not.

In (3), the asterisk placed next to the value 1 indicates the special significance of the value 1. The inner meanings attached to the symbols 1 and 2 may be "truth" and "falsity" respectively, and (3) expresses some intuitions suggested by an analysis of implication.

J. Łukasiewicz [2] and E. Post [3] have formed many-valued logics by

---

[1] See [8].

[2] See [5], [7].

[3] See [9].

extending the method. For example, Łukasiewicz proposes the table

$p \to q$	1	2	3	q
*1	1	2	3	
2	1	1	1	
3	1	3	1	
p				

(4)

as the table for the implication of his "ternary logic of implication."[1] To decide whether (2) is a theorem of this logic, all possible permutations (repeated elements being allowed) of the numbers 1, 2, 3 are substituted for p, q, r, and the resulting expression contracted according to table (4). If (2) contracts to 1 for all substitutions, then (2) is a theorem of the ternary logic; if (2) contracts into 2 or 3 for at least one substitution, then (2) is not a theorem of that logic.

The author has succeeded in proving that, for every $n > m$ and any m-valued truth-table, there is an n-valued truth-table such that the two corresponding m-valued and n-valued logics are the same.[2,3] The author has discussed also the problems of mutual relationships between logics formed by different truth-tables with finite and infinite numbers of values.[4] But no method seems to be known for deciding whether two given truth-tables form the same set of theorems or not (i.e. are "equivalent" or not).

In this note a method is presented by which two truth-tables can, in certain cases, be proved equivalent. The method is demonstrated for the case of four particular truth-tables (called A_2, A_3, A_4 and A_5 below), but generalisation to cases of similar type presents no difficulty.

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[1] See [6]. Łukasiewicz writes 0 for 2, $\frac{1}{2}$ for 3, 1 for 1. "Possibility" is the inner meaning of $\frac{1}{2}$.

[2] See [3].

[3] To justify calling two logics generated by two truth-tables with different numbers of values the "same," it will be noted that many-valued logics are basically formalisations for which certain formulae are called "theorems" and others not. In the sense of this paper two formalisations are the "same" logics, irrespective of possible truth-table interpretations, if the set Ω of possible well-formed formulae is the same for the two formalisations and if the classification of each element of Ω into a "theorem" or "non-theorem" is the same for the two formalisations. (Cf. e.g. [10], [1], [7].)

[4] See [2], [4].

First. the equivalence of

$A_5 \equiv$

$p \to q$	1	2	3	4	5	q
*1	1	2	3	4	5	
2	1	1	3	3	5	
3	1	2	1	2	5	
4	1	1	1	1	5	
5	1	2	3	4	1	
p						

and

$A_2 \equiv$

$p \to q$	1	2	q
*1	1	2	
2	1	1	
p			

will be proved. To do this it will be shown that

(i) every non-theorem of A_5 is not a theorem of A_2,

(ii) every non-theorem of A_2 is not a theorem of A_5.

(i) If an implicational formula α is not a theorem of A_5, there is a substitution of the values 1, 2, 3, 4, 5 for its variables which makes α contract to a value different from 1. The variables of α may be classified into five categories I, II, III, IV, V such that, when the substitution referred to above is made, the value 1 is substituted for the variables in category I, the value 2 is substituted for the variables in category II, and so on. (Some categories may be missing.) Suppose by this substitution α contracted to 5; then the process may be written

$$\alpha\,(\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = 5. \tag{5}$$

Now it will be noticed that the values 1, 2, 3, 4 in A_5 are such that, if value 1 is written in place of values 1, 2, 3, 4 (without changing the value 5), there results the table

$p \to q$	1	1	1	1	5	q
1	1	1	1	1	5	
1	1	1	1	1	5	
1	1	1	1	1	5	
1	1	1	1	1	5	
5	1	1	1	1	1	
p						

(6)

in which the first four columns and the first four rows are repetitions of the first column and the first row respectively (the question of asterisks being disregarded). In such cases, the values of 1, 2, 3, 4 will be called *similar*, written

$$1 \sim 2 \sim 3 \sim 4\,; \quad 5 \sim 5.$$

Table (6) may then be rewritten as

$p \to q$	1	5	q
1	1	5	
5	1	1	
p			

(7)

Thus, substitution of value 1 instead of values 2, 3, 4, for the variables of categories II, III, IV of α gives, by table (7),

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/1},\ \mathrm{III/1},\ \mathrm{IV/I},\ \mathrm{V/5}) = 5. \tag{8}$$

Comparison of A_2 and table (7) gives at once

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/1},\ \mathrm{III/1},\ \mathrm{IV/1},\ \mathrm{V/2}) = 2 \tag{a}$$

by A_2. In other words, α is not a theorem of A_2.

Similarly, if, according to A_5,

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/2},\ \mathrm{III/3},\ \mathrm{IV/4},\ \mathrm{V/5}) = 4 \tag{9}$$

then

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/1},\ \mathrm{III/4},\ \mathrm{IV/4},\ \mathrm{V/1}) = 4 \tag{10}$$

according to

$p \to q$	1	4	q
1	1	4	
4	1	1	
p			

(11)

since $1 \sim 2 \sim 5$; $3 \sim 4$ in A_5.

Comparison of A_2 and table (11) gives at once

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/1},\ \mathrm{III/2},\ \mathrm{IV/2},\ \mathrm{V/1}) = 2 \tag{b}$$

by A_2.

Further, if

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/2},\ \mathrm{III/3},\ \mathrm{IV/4},\ \mathrm{V/5}) = 3 \tag{12}$$

by A_5, then

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/1},\ \mathrm{III/3},\ \mathrm{IV/3},\ \mathrm{V/1}) = 3 \tag{13}$$

according to

$p \to q$	1	3	q
1	1	3	
3	1	1	
p			

(14)

since $1 \sim 2 \sim 5$; $3 \sim 4$ in A_5, and hence

$$\alpha\,(\mathrm{I/1},\ \mathrm{II/1},\ \mathrm{III/2},\ \mathrm{IV/2},\ \mathrm{V/1}) = 2 \tag{c}$$

by A_2.

Finally, if

$$\alpha\,(\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = 2 \tag{15}$$

by A_5, then

$$\alpha\,(\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/1,\ \mathrm{IV}/2,\ \mathrm{V}/1) = 2 \tag{d}$$

by A_2, since $1 \sim 3 \sim 5$; $2 \sim 4$ in A_5.

This completes the proof of (i).

(ii) If, now, α is not a theorem of A_2, then must

$$\alpha\,(\mathrm{I}/1,\ \mathrm{II}/2) = 2 \tag{e}$$

according to A_2, and hence

$$\alpha\,(\mathrm{I}/1,\ \mathrm{II}/5) = 5 \tag{16}$$

according to table (7), and consequently also according to A_5.

This completes the proof of (ii).

Thus A_2 and A_5 are equivalent. Moreover, the tables

$A_3 \equiv$

$p \to q$	1	2	3	q
*1	1	2	3	
2	1	1	3	
3	1	2	1	
p				

and $A_4 \equiv$

$p \to q$	1	2	3	4	q
*1	1	2	3	4	
2	1	1	3	3	
3	1	2	1	2	
4	1	1	1	1	
p					

are equivalent to A_2 and A_5, since every theorem of A_5 is also a theorem of A_4 and A_3, every theorem of A_4 and A_3 is a theorem of A_2, and A_2 and A_5 are equivalent. The result for A_3 has been obtained by the author by a different and more complicated method.[1]

It may be of interest to note that A_5, A_4, and A_3 are such that if according to them any formula takes a value different from 1 then it takes all the values of the particular table other than 1.

To prove this for A_5, it will be seen by the above that if

$$\alpha\,(\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = k \qquad (k = 2, 3, 4, 5)$$

then there are values

$$a_1\ a_2\ a_3\ a_4\ a_5 \qquad (a_i = 1, 2, \qquad i = 1, 2, 3, 4, 5)$$

such that

$$\alpha\,(\mathrm{I}/a_1,\ \mathrm{II}/a_2,\ \mathrm{III}/a_3,\ \mathrm{IV}/a_4,\ \mathrm{V}/a_5) = 2 \tag{f}$$

according to A_2.

Comparing A_2 with (7), (11), and (14), it will be seen that by substituting $b_i = 1$ for the variables of those categories for which $a_i = 1$ in (f), and $b_i = h$, where h is any one of the values 2, 3, 4, 5, for the variables of those categories for which $a_i = 2$ in (f), then

[1] See [3].

$$\alpha\,(\mathrm{I}/b_1,\ \mathrm{II}/b_2,\ \mathrm{III}/b_3,\ \mathrm{IV}/b_4,\ \mathrm{V}/b_5) = h.$$

Similar procedures can be applied to A_3 and A_4. For example, the formula

$$\alpha \equiv (\,(\,(s \to p) \to (q \to r)\,) \to s) \to (\,(\,(q \to q) \to (p \to p) \to r)\,)$$

takes the value 4 according to A_5 if 2, 3, 4, 5 are substituted for p, q, r, s respectively. Hence, to make α contract into 2 according to A_2, then by (c) the substitution

$$\alpha\,(p/1,\ q/2,\ r/2,\ s/1) = 2$$

is made. Therefore to make α contract into 5, say, according to A_5, the substitution

$$\alpha\,(p/1,\ q/5,\ r/5,\ s/1) = 5$$

is made. This may be verified directly by substitution in α and evaluation by A_5.

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REFERENCES.

1. A. Heyting, "Die formalen Regeln der intuitionistischen Logik," *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse*, 1930.

2. J. Kalicki, "On Tarski's Matrix Method," *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, XLI (1948), Classe III, 130-142.

3. J. Kalicki, "Note on Truth-Tables," *Journal of Symbolic Logic*, 15 (1950), 174-181.

4. J. Kalicki, "A Test for the Existence of Tautologies according to Many-Valued Truth-Tables," *Journal of Symbolic Logic*, 15 (1950), 182-184.

5. J. Łukasiewicz, A Review, *Ruch Filozoficzny*, vol. 5, No. 9 (1920).

6. J. Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls," *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, XXIII (1930).

7. J. Łukasiewicz and A. Tarski, "Untersuchungen über den Aussagenkalkül," *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, XXIII (1930).

8. C. S. Peirce, "On the Algebra of Logic; A Contribution to the Philosophy of Notation," *American J. of Math.*, VII (1885).

9. E. L. Post, "Introduction to a general theory of elementary propositions," *American J. of Math.*, XLIII (1921).

10. J. B. Rosser and A. R. Turquette, "Axiom schemes for m-valued propositional calculi," *Journal of Symbolic Logic*, X (1945).

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Integral Functions with Gap Power Series

By P. Erdős and A. J. Macintyre

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1. Let

$$f(z) = \sum_0^\infty a_n z^{\lambda_n} \tag{1}$$

be an integral function, λ_n being a strictly increasing sequence of non-negative integers. We shall use the notations

$$M(r) = \max_{|z|=r} |f(z)|, \; m(r) = \min_{|z|=r} |f(z)|,$$
$$\mu(r) = \max_{n=0,1,2,\ldots} |a_n| r^n,$$

describing $M(r)$ as the maximum modulus, $m(r)$ as the minimum modulus and $\mu(r)$ as the maximum term of $f(z)$.

The present paper is a development of a remark by Pólya (*Math. Zeit.*, 29 (1929), 549-640, last sentence of the paper) that if

$$\lim \frac{\log(\lambda_{n+1} - \lambda_n)}{\log \lambda_n} > \tfrac{1}{2} \tag{2}$$

then

$$\overline{\lim_{r\to\infty}} \frac{m(r)}{M(r)} = \overline{\lim_{r\to\infty}} \frac{\mu(r)}{M(r)} = 1. \tag{3}$$

Our first result is

Theorem 1.

If

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < \infty, \tag{4}$$

then (3) *holds.*

Theorem 1 is clearly a sharpened form of Pólya's result, for from (2) it evidently follows that for sufficiently large n

$$\lambda_{n+1} - \lambda_n > \lambda_n^{\frac{1}{2}+\epsilon} > n^{1+\delta} \text{ for some positive } \epsilon \text{ and } \delta.$$

Theorem 1 is best possible, as is shown by our next result.

Theorem 2.

If

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} = \infty, \tag{4}$$

then there exists an integral function of the form (1) *such that*

$$\overline{\lim_{r\to\infty}}\ \frac{\mu(r)}{M(r)} \leqq \tfrac{1}{2},\quad \overline{\lim_{r\to\infty}}\ \frac{m(r)}{M(r)} \leqq \tfrac{1}{2}. \tag{6}$$

We generalise these theorems in two ways. First, relaxing the gap hypothesis we have

THEOREM 3.

If for a positive integer h

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} < \infty \tag{7}$$

then

$$\overline{\lim_{r\to\infty}}\ \frac{\mu(r)}{M(r)} \geqq \frac{1}{2h-1}\,; \tag{8}$$

but if

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} = \infty \tag{9}$$

for every h, *then there exists an integral function of the form* (1) *such that*

$$\lim_{r\to\infty} \frac{\mu(r)}{M(r)} = \lim_{r\to\infty} \frac{m(r)}{M(r)} = 0. \tag{10}$$

The conjecture that under condition (7) we could derive

$$\overline{\lim_{r\to\infty}}\ \frac{m(r)}{M(r)} > 0 \tag{11}$$

is disproved trivially by the example

$$\sum_0^{\infty} z^{n^3}/(n^3)! + \sum_0^{\infty} z^{n^3+1}/(n^3+1)!\,.$$

Our second generalisation relaxes the gap condition of Theorem I in a different way, but imposes in addition a condition on the order of the function. We have

THEOREM 4.

If as n *tends to infinity*

$$\sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = o(\log \lambda_n), \tag{12}$$

and the function $f(z)$ *is of finite order, or if*

$$\sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = O(\log \lambda_n), \tag{13}$$

and $f(z)$ *is of zero order, then* (2) *holds.*

This theorem cannot be materially strengthened since the example

constructed for Theorem 2 will be of finite order if

$$\lim_{n\to\infty} \frac{1}{\log \lambda_n} \sum_{k=0}^{n} \frac{1}{\lambda_{k+1}-\lambda_k} > 0$$

and of zero order if

$$\lim_{n\to\infty} \frac{1}{\log \lambda_n} \sum_{k=0}^{n} \frac{1}{\lambda_{k+1}-\lambda_k} = \infty .$$

2. *Proof of Theorem* 1. To prove the theorem we need an elementary inequality. If $\epsilon_0 + \epsilon_1 + \epsilon_2 + \ldots$ is a convergent series of non-negative numbers and if a sequence δ_n is defined by

$$\delta_n = \max_{i<n<j} \frac{1}{(j-i+1)^{\frac{3}{2}}} \sum_{v=i}^{j} \epsilon_v, \tag{14}$$

then

$$\sum_0^\infty \delta_n \leqq (1 + 2 \sum_{n=2}^{\infty} n^{-\frac{3}{2}}) \sum_0^\infty \epsilon_v. \tag{15}$$

We have

$$\sum_0^\infty \delta_n = \sum_0^\infty \sum_0^\infty A_{v,n}\epsilon_v,$$

where $A_{v,n} = (j_n - i_n + 1)^{-3/2}$ or zero, as v falls in $i_n \leqq v \leqq j_n$ or not, i_n, j_n being the values of i, j for which the maximum in (14) is attained. Since $i_n \leqq n \leqq j_n$ also it follows that $j_n - i_n \geqq |\, v-n \,|$. Consequently

$$\sum_0^\infty \delta_n \leqq \sum_0^\infty \sum_0^\infty \frac{\epsilon_v}{(\,|\, v-n \,|\, + 1)^{3/2}}$$

$$\leqq (1 + 2\sum_0^\infty n^{-3/2}) \sum_0^\infty \epsilon^n.$$

We now assume (4) and set

$$\epsilon_n = 1/(\lambda_{n+1} - \lambda_n). \tag{16}$$

Defining δ_n as in (14), we have $\sum_0^\infty \delta_n < \infty$ by (15). Let c_n be a sequence of positive numbers tending to infinity so slowly that

$$\sum_0^\infty c_n \delta_n < \infty . \tag{17}$$

Now let $A_n \leqq |\, z \,| \leqq A_{n+1}$, $n = 0, 1, 2, \ldots$, be the sequence of intervals in which a single term $a_k z^{\lambda_k}$ remains the maximum term. k will depend on n and increases with n, but we need not express this dependence in our notation. From (17) we have $\prod_0^\infty (1 + 2c_k\delta_k)^2 < \infty$, and hence there exist arbitrarily large values of n such that

$$A_{n+1}/A_n > (1 + 2\,c_k\,\delta_k)^2. \tag{18}$$

We understand by n such a value and by k the associated integer. Since $a_k z^{\lambda_k}$ is the maximum term for $A_n \leqq |z| \leqq A_{n+1}$, we have

$$\begin{aligned} |a_v| &\leqq |a_k| A_n^{\lambda_k - \lambda_v} && (v < k) \\ |a_v| &\leqq |a_k| A_{n+1}^{-(\lambda_v - \lambda_k)} && (v > k). \end{aligned} \tag{19}$$

Using these inequalities with $r = |z| = (A_n A_{n+1})^{\frac{1}{2}}$, we have

$$\begin{aligned} |a_v| r^{\lambda_v} &\leqq |a_k| r^{\lambda_k} (A_n/A_{n+1})^{\frac{1}{2}(\lambda_k - \lambda_v)} \\ &\leqq |a_k| r^{\lambda_k} (1 + 2c_k\delta_k)^{-(\lambda_k - \lambda_v)} && (v < k), \\ |a_v| r^{\lambda_v} &\leqq |a_k| r^{\lambda_k} (1 + 2c_k\delta_k)^{-(\lambda_v - \lambda_k)} && (v > k). \end{aligned} \tag{20}$$

But by the definition of δ_n and the inequality of the harmonic and arithmetic means,

$$\begin{aligned} \delta_k &\geqq \left(\frac{1}{\lambda_{v+1} - \lambda_v} + \frac{1}{\lambda_{v+2} - \lambda_{v+1}} + \ldots + \frac{1}{\lambda_k - \lambda_{k-1}} \right) (k - v)^{-\frac{1}{2}} \\ &\geqq \frac{1}{(k-v)^{\frac{1}{2}}} \left(\frac{k - v}{\lambda_k - \lambda_v} \right) = \frac{(k-v)^{\frac{1}{2}}}{\lambda_k - \lambda_v} \qquad (v < k). \end{aligned} \tag{21}$$

Consequently

$$(1 + 2c_k\gamma_k)^{-(\lambda_k - \lambda_v)} \leqq e^{-c_k(k-v)^{\frac{1}{2}}} \qquad (v < k). \tag{22}$$

From this and a similar inequality when $v > k$, it follows from (20) that as $n \to \infty$ (and so $k \to \infty$, $r \to \infty$, $c_n \to \infty$)

$$\sum_0^{k-1} |a_v| r^{\lambda_v} + \sum_{k+1}^{\infty} |a_v| r^{\lambda_v} = o(|a_k| r^{\lambda_k}). \tag{23}$$

From this follow first the second and then evidently the first statement of (3).

3. *Proof of Theorem* 2. Now suppose that $\sum_0^{\infty} 1/(\lambda_{n+1} - \lambda_n)$ diverges. We choose the coefficients a_n by the following rules.

$$a_0 = 1, \qquad a_n = a_{n+1} A_n^{-(\lambda_n - \lambda_{n-1})}, \tag{24}$$

where

$$A_n = \prod_{v=0}^{n-1} \left(1 + \frac{\epsilon_v}{\lambda_v - \lambda_{v-1}} \right), \qquad A_0 = 1, \qquad A_1 = \left(1 + \frac{1}{\lambda_0 + 1} \right) \tag{25}$$

and ϵ_n is a a sequence of positive numbers tending to zero and such that $\Sigma \epsilon_n/(\lambda_{n+1} - \lambda_n)$ diverges.

Evidently $A_n \to \infty$ and $f(z) = \sum_0^{\infty} a_n z^{\lambda_n}$ is an integral function.

Since

$$\frac{a_{n+1} r^{\lambda_{n+1}}}{a_n r^{\lambda_n}} = \frac{r^{\lambda_{n+1} - \lambda_n}}{A_{n+1}^{\lambda_{n+1} - \lambda_n}}, \tag{26}$$

the maximum term $\mu(r)$ is $a_n r^{\lambda_n}$ for

$$A_n \leqq r \leqq A_{n+1}. \tag{27}$$

Clearly

$$M(r) = \sum_0^\infty a_n r^{\lambda_n} > a_n r^{\lambda_n} + a_{n+1} r^{\lambda_n + 1}. \tag{28}$$

Now for $A_n \leqq r \leqq A_{n+1}$ we have

$$\begin{aligned} \frac{a_{n+1} r^{\lambda_{n+1}}}{a_n r^{\lambda_n}} = \left(\frac{r}{A_{n+1}}\right)^{\lambda_{n+1} - \lambda_n} &\geqq \left(\frac{A_n}{A_{n+1}}\right)^{\lambda_{n+1} - \lambda_n} \\ &= \left(1 + \frac{\epsilon_n}{\lambda_{n+1} - \lambda_n}\right)^{-(\lambda_{n+1} - \lambda_n)} > e^{-\epsilon_n}, \end{aligned} \tag{29}$$

and it follows that $M(r) > (2 - \epsilon)\,\mu(r)$ for all sufficiently large r.

This proves the first inequality of (6). To establish the second we argue as follows. With $A_n \leqq r \leqq A_{n+1}$ and $z = re^{\pi i/(\lambda_{n+1} - \lambda_n)}$ we have, for n sufficiently large,

$$\begin{aligned} |f(z)| &\leqq M(r) - a_n r^{\lambda_n} - a_{n+1} r^{\lambda_{n+1}} + (a_n r^{\lambda_n} - a_{n+1} r^{\lambda_{n+1}}) \\ &= M(r) - 2a_{n+1} r^{\lambda_{n+1}} \leqq M(r) - (2 - \epsilon)\,\mu(r). \end{aligned} \tag{30}$$

If $\mu(r) \geqq \frac{1}{4} M(r)$, it follows that $m(r) \leqq (\frac{1}{2} + \epsilon) M(r)$.

If $\mu(r) < \frac{1}{4} M(r)$ we argue differently. We use the relations

$$\{m(r)\}^2 \leqq \{M_2(r)\}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta = \sum_0^\infty a_n^2 r^{2\lambda_n}, \tag{31}$$

which lead to

$$\begin{aligned} \{M(r)\}^2 &\geqq \sum_0^\infty a_\nu^2 r^{2\lambda_\nu} + \sum_0^\infty a_\nu r^{\lambda_\nu} \{f(r) - a_\nu r^{\lambda_\nu}\} \\ &\geqq \{M_2(r)\}^2 + \sum_0^\infty a_\nu r^{\lambda_\nu} \{f(r) - \tfrac{1}{4} f(r)\} \end{aligned} \tag{32}$$

and

$$\{m(r)\}^2 \leqq \{M_2(r)\}^2 \leqq \tfrac{1}{4} \{M(r)\}^2. \tag{33}$$

4. *Proof of Theorem* 3.

Suppose now that

$$\sum_{n=0}^\infty \frac{1}{\lambda_{n+h} - \lambda_n} < \infty, \tag{34}$$

where h is a positive integer greater than unity.

Defining δ_n as in (14) with $\epsilon_n = (\lambda_{n+h} - \lambda_n)^{-1}$ and choosing $c_n > 0$ so that $c_n \to +\infty$ and $\Sigma c_n \delta_n < \infty$, and again taking $A_n \leqq |z| < A_{n+1}$

to be the sequence of intervals in which a single term, say $a_k z^{\lambda_k}$, is the maximum term, we must have arbitrarily large values of n such that $A_{n+1}/A_n > (1 + 2c_k\delta_k)^2$, that is condition (18). With such values of n and associated k we still have (19) and (20), but we can no longer expect such a good result as (21) or its consequences (22) and (23). For $r = (A_n A_{n+1})^{\frac{1}{2}}$ and v "near" to k we can only say

$$| a_v | r^{\lambda_v} \leqq | a_k | r^{\lambda_k} \qquad (k - h < v < k + h). \tag{35}$$

For values of v which are not "too near" k we can give an analogue of (21) valid for $k - ph < v \leqq k - (p-1)h$, $\quad p = 2, 3, \ldots$, in

$$\delta_k \geqq \left(\frac{1}{\lambda_{k-(p-2)h} - \lambda_{k-(p-1)h}} + \ldots + \frac{1}{\lambda_{k-h} - \lambda_{k-2h}} + \frac{1}{\lambda_k - \lambda_{k-h}} \right) \frac{1}{(ph)^{\frac{1}{2}}}$$

$$\geqq \frac{(p-1)^2}{\lambda_k - \lambda_{k-(p-1)h}} \frac{1}{(ph)^{\frac{1}{2}}} \geqq \frac{p^{\frac{1}{2}}}{4h^{\frac{1}{2}}(\lambda_k - \lambda_v)}$$

$$\geqq \frac{(k-v)^{\frac{1}{2}}}{4h^2(\lambda_k - \lambda_v)}.$$

Consequently

$$(1 + 2c_k\delta_k)^{-(\lambda_k - \lambda_v)} \leqq e^{-c_k (k-v)^{\frac{1}{2}}/4h^2}.$$

From this and the similar inequalities with $v > k + h$ we have, as $n \to \infty$, the result

$$\sum_0^{k-h} | a_v | r^{\lambda_v} + \sum_{k+h}^{\infty} | a_v | r^{\lambda_v} = o(| a_k | r^{\lambda_k}), \tag{36}$$

and consequently with (35) we deduce

$$\underline{\lim}\ M(r)/\mu(r) \leqq (2h - 1)$$

or

$$\overline{\lim}\ \mu(r)/M(r) \geqq 1/(2h - 1),$$

which constitutes the first part of Theorem 3.

Now suppose that for some integer $h > 1$

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} = \infty.$$

Then evidently one of the series

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{nh+h+k} - \lambda_{nh+k}} \qquad (k = 0, 1, \ldots, h-1) \tag{37}$$

must diverge. There will be no loss of generality in supposing that the series with $k = 0$ diverges. We now, as in the proof of Theorem 2, define the series

$$f^*(z) = \sum_0^{\infty} a_n z^{\lambda^* n}, \qquad \lambda_n^* = \lambda_{nh} \tag{38}$$

with the properties that

$$\text{(i)}\quad \mu^*(r) = a_n^* r^{\lambda^* n} \qquad \text{(ii)}\quad a_{n+1}^* r^{\lambda^* n+1} \geqq (1-\epsilon)\, a_n^* r^{\lambda^* n} \tag{39}$$

$$\text{for } A_n^* \leqq r \leqq A_{n+1}^*, \qquad n > n(\epsilon),$$

where $\mu^*(r)$ is the maximum term of $f^*(z)$ and A_n^* is defined from the sequence λ_n^* as A_n is defined from λ_n in (25). Let us now define $f(z) = \sum_0^\infty a_n z^{\lambda_n}$ by the conditions

$$a_{nh} = a_n^*,\ a_{nh+k} = a_n^* A_{n+h}^{-(\lambda_{nh+h} - \lambda_{nh})} \qquad (k = 1, 2, \ldots, h-1). \tag{40}$$

Then evidently for $A_n^* \leqq r \leqq A_{n+1}^*$ we shall have

$$a_{nh} r^{\lambda_{nh}} \geqq a_{nh+1} r^{\lambda_{nh}+1} \geqq \ldots \geqq a_{nh+h} r^{\lambda_{nh}+h}, \tag{41}$$

and $\mu(r)$ for the function $f(z)$ will be $a_{nh} r^{\lambda_{nh}}$, so that

$$M(r) = f(r) > (h+1-\epsilon)\,\mu(r) \qquad [r > r(\epsilon)]. \tag{42}$$

We approximate $m(r)$ by using

$$\{m(r)\}^2 \leqq \{M_2(r)\}^2 = \sum_0^\infty a_\nu^2 r^{2\lambda_\nu}. \tag{43}$$

Clearly

$$\{M(r)\}^2 = \sum_0^\infty a_\nu^2 r^{2\lambda_\nu} + \sum_0^\infty a_\nu r^{\lambda_\nu}\{M(r) - a_\nu r^{\lambda_\nu}\} \tag{44}$$

$$\geqq \{M_2(r)\}^2 + \{M(r)\}^2 - (h+1-\epsilon)^{-1}\{M(r)\}^2,$$

from which

$$m(r) \leqq M_2(r) \leqq (h+1-\epsilon)^{-\frac{1}{2}} M(r) \tag{45}$$

follows.

This does not quite complete the proof of Theorem 3 since $(h+1-\epsilon)^{-1}$ and $(h+1-\epsilon)^{-\frac{1}{2}}$, although arbitrarily small, are not zero. However we should only have to choose λ_n^* to be a subsequence of λ_n such that the interval $\lambda_n^* \leqq \lambda \leqq \lambda_{n+1}^*$ contains a number of λ_n increasing with λ_n^* but that $\Sigma\,(\lambda_{n+1}^* - \lambda_n^*)^{-1}$ diverges. It does not seem necessary to enumerate the details.

5. *Proof of Theorem* 4.

Given an increasing sequence of integers λ_n, let us first try to construct an integral function $\sum_0^\infty c_n x^{\lambda_n}$ with positive coefficients such that each term is in turn the maximum term and greatly exceeds in

value the rest of the series. More precisely let $\delta > 0$ be a small prescribed number and let us choose the c_n in such a way that for a certain increasing sequence A_n of positive numbers the following conditions hold for all N. For $x = A_N$ we require that

$$\begin{aligned} c_{N+1}\, x^{\lambda_N+1} &= \delta c_N\, x^{\lambda_N} \\ c_{N-1}\, x^{\lambda_N-1} &= \delta c_N\, x^{\lambda_N}. \end{aligned} \tag{46}$$

In this case we shall have, for $n > N$ and $x = A_N$,

$$c_{n+1}\, x^{\lambda_n+1} = \delta c_n\, x^{\lambda_n} \tag{47}$$

and consequently, for $x = A_N < A_n$,

$$c_{n+1}\, x^{\lambda_n+1} \leqq \delta c_n x^{\lambda_n}. \tag{48}$$

So for $x = A_N$, $p > 0$,

$$\begin{aligned} c_{N+p}\, x^{\lambda_N+p} &\leqq \delta^p\, c_N\, x^{\lambda_N} \\ \sum_{N+1}^{\infty} c_n\, x^{\lambda_n} &\leqq \frac{\delta}{1-\delta} c_N\, x^{\lambda_N}. \end{aligned} \tag{49}$$

Similarly, for $x = A_N$,

$$\sum_{0}^{N+1} c_n\, x^{\lambda_n} \leqq \frac{\delta}{1-\delta} c_N\, x^{\lambda_N}. \tag{50}$$

We must now consider whether our conditions are possible.

(46) requires that

$$\begin{aligned} c_{N+1} &= \delta c_N / A_N^{\lambda_{N+1}-\lambda_N} \\ c_N &= \delta c_{N+1}\, A_{N+1}^{\lambda_{N+1}-\lambda_N}. \end{aligned} \tag{51}$$

Eliminating c_N and c_{N+1}, we see that

$$A_{N+1}/A_N = \delta^{-2/(\lambda_{N+1}-\lambda_N)} = K^{1/(\lambda_{N+1}-\lambda_N)} \qquad (K > 1). \tag{52}$$

This defines the sequence A_n if we take $A_0 = 1$, and shows that it is increasing. With $c_1 = 1$ the sequence c_n is also defined, for the two conditions of (46) are now equivalent. The function $\sum_1^{\infty} c_n\, x^{\lambda_n}$ will be an integral function if A_n tends to infinity. Since

$$\log A_n = \log K \left\{ \frac{1}{\lambda_1 - \lambda_0} + \frac{1}{\lambda_2 - \lambda_1} + \ldots + \frac{1}{\lambda_n - \lambda_{n-1}} \right\}, \tag{53}$$

this condition requires the divergence of $\sum_1^{\infty} 1/(\lambda_{n+1} - \lambda_n)$.

The property of domination by single terms expressed by (49) and (50) will be carried over to the integral function $\sum_0^\infty a_n z^{\lambda_n}$ if we can assert that

$$\sum_0^\infty a_n z^{\lambda_n}/c_n \tag{54}$$

is an integral function. If we make the hypothesis that $\sum_0^\infty a_n z^{\lambda_n}$ is of finite order then $|a_n| < \lambda_n^{-\alpha\lambda_n}$ for sufficiently large n and some positive α. To ensure that (54) does define an integral function we shall require to prove that for arbitrary $\epsilon > 0$ and sufficiently large n.

$$c_n > \lambda_n^{-\epsilon\lambda_n}. \tag{55}$$

This is equivalent to $\log c_n > -\epsilon\lambda_n \log \lambda_n$
and since

$$\log c_n = n \log \delta - \sum_{\nu=0}^{n-1} (\lambda_{\nu+1} - \lambda_\nu) \log A_\nu \tag{56}$$

this will follow from

$$\log A_n = o(\log \lambda_n) \tag{57}$$

or

$$\sum_1^n \frac{1}{\lambda_\nu - \lambda_{\nu-1}} = o(\log \lambda_n). \tag{58}$$

Now if we assume that $\sum_0^\infty a_n z^{\lambda_n}/c_n$ is an integral function it will follow that for sufficiently large values of z, say $z = R$, the maximum term of this function will occur with $n = N$ arbitrarily large. We shall have

$$|a_n| R^{\lambda_n}/c_n \leqq |a_N| R^{\lambda_N}/c_N.$$

$$\frac{|a_n| R^{\lambda_n}}{|a_N| R^{\lambda_N}} \leqq \frac{c_n}{c_N}$$

$$\frac{|a_n| (RA_N)^{\lambda_n}}{|a_N| (RA_N)^{\lambda_N}} \leqq \frac{c_n (A_N)^{\lambda_n}}{c_N (A_N)^{\lambda_N}}.$$

Thus the dominance expressed by (49) and (50) of a single term for $\Sigma c_n z^{\lambda_n}$ holds also for the function $\Sigma a_n z^{\lambda}$ with $|z| = RA_N$. Since δ may be chosen arbitrarily small Theorem 4 is proved for functions of finite order. If $\Sigma a_n x^{\lambda_n}$ is assumed to be of zero order we only require that $c_n > \lambda_n^{-h\lambda_n}$ for some positive h, and this clearly follows from (13).

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Integral Equations Satisfied by Lamé-Wangerin Functions

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Summary.

Integral equations are obtained with nuclei $(1 - zt/a)^{2n}$ and $(z-t)^{2n}$ which are satisfied by characteristic solutions of the transformed Lamé-Wangerin equation of order n, and each of the two characteristic solutions is expressed in terms of the other by a contour integral.

1. *Introduction.*

Lamé-Wangerin functions are the solutions of Lamé's differential equation of order n when n is half of an odd integer. If $n = m + \frac{1}{2}$, where m is an integer, Lamé's equation in algebraic form is

$$4x(x-1)(x-a)\frac{d^2u}{dx^2} + 2\{3x^2 - 2(a+1)x + a\}\frac{du}{dx}$$

$$-\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})x + h\}u = 0 \qquad (1.1)$$

and there is no loss of generality in taking the finite singularities as 0, 1 and a.

Writing $$x = (z^2 - a)^2 / 4z(z-1)(z-a)$$

and $$u(x) = \{z(z-1)(z-a)\}^{(2m+1)/4} v(z),$$

we have the transformed equation considered by Halphen and others:[1]

$$z(z-1)(z-a)\frac{d^2v}{dz^2} - m\left\{3z^2 - 2(a+1)z + a\right\}\frac{dv}{dz}$$

$$+\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})z - h - (m+\tfrac{1}{2})^2(a+1)\}v = 0. \qquad (1.2)$$

If h has one of a set of $m+1$ characteristic values this equation has two solutions which are polynomials in z. I shall denote them by

$$v_1(z) = \sum_{r=0}^{m} c_r a^{m+\frac{1}{2}-r} z^r,$$

$$v_2(z) = \sum_{r=0}^{m} c_r z^{2m+1-r}, \qquad (1.3)$$

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[1] G. H. Halphen, *Traité des Fonctions Elliptiques* (Paris, 1888), t.2, p. 471. Whittaker and Watson, *Modern Analysis* (Cambridge, 1920, 3rd ed.), §§ 23-7.

the coefficients c_ν being the same in the two solutions and $c_0 = 1$. Substitution of either of these solutions in (1.2) leads to the recurrence relations

$$(\nu - m)(\nu + 1)\, c_{\nu+1} - \{h + (m + \tfrac{1}{2} - \nu)^2 (a + 1)\} c_\nu + a(m + 1 - \nu)(2m + 2 - \nu)\, c_{\nu-1} = 0 \qquad (1.4)$$

for $\nu = 0, 1, \ldots, m$.

The first m relations determine the coefficients c_1 to c_m, the coefficient c_ν being a polynomial of degree ν in h; and putting $\nu = m$ we have

$$-\{h + (a + 1)/4\}\, c_m + (m + 2) a\, c_{m-1} = 0. \qquad (1.5)$$

The expression in (1.5) is a polynomial of degree $m + 1$ in h determining $m + 1$ characteristic values of h which are real and distinct.

We have also

$$v_2(z) = \frac{z^{2m+1}}{a^{m+\frac{1}{2}}} v_1(a/z), \qquad v_1(z) = \frac{z^{2m+1}}{a^{m+\frac{1}{2}}} v_2(a/z). \qquad (1.6)$$

2. *Types of integral equation.*

The differential equation (1.2) in Riemannian form is

$$v(z) = P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & -m & z \\ m+1 & m+1 & m+1 & -2m-1 & \end{matrix}\right\}. \qquad (2.1)$$

A general theorem [1] on integral equations associated with differential equations of this type shows that characteristic and certain other solutions of (1.2) satisfy an integral equation whose nucleus is the hypergeometric function

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & -m & zt/a \\ m+1 & 2m+1 & -2m-1 & \end{matrix}\right\}.$$

We may take this function as being $(1 - zt/a)^{2m+1}$ and consider integrals of the type

$$u(z) = \frac{\lambda}{2\pi i} \int_C \left(1 - \frac{zt}{a}\right)^{2m+1} \frac{v(t)\, dt}{\{t(t-1)(t-a)\}^{m+1}} \qquad (2.2)$$

where C is a closed curve encircling one or more of the singularities $t = 0$, $t = 1$, $t = a$.

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[1] C. G. Lambé and D. R. Ward, *Quart. J. of Math.* (Oxford), 5 (1934), 81 and A. Erdélyi, *Quart. J. Math.* (Oxford), 13 (1942), 107.

The general theorem shows that if $v(z)$ is a solution of (1.2) corresponding to a characteristic value h_r of h, then $u(z)$ is also a solution of (1.2) for the same value of h.

From the relations (1.6) it follows that if z is replaced by a/z,

$$u(a/z) = a^{m+\frac{1}{2}} z^{-2m-1} w(z),$$

where $w(z)$ is a solution of (1.2). Hence integrals of the type

$$w(z) = \frac{\lambda}{2\pi i} \int_C (z-t)^{2m+1} \frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}} \tag{2.3}$$

are also solutions of (1.2).

3. *Contours enclosing one singularity.*

If C is a contour enclosing only the singularity $t = 0$, the value of the integral (2.2) is the residue at $t = 0$. Hence $u(z)$ must be of degree m in z and therefore must be a multiple of $v_1(z)$.

If C encloses only the singularity $t = 1$, then, since powers of $(t-1)$ higher than the m^{th} in the expansion of

$$(1 - zt/a)^{2m+1} = \{(1 - z/a) - (t-1)\,z/a\}^{2m+1}$$

give zero residue at $t = 1$, it follows that $(z-a)^{m+1}$ is a factor of $u(z)$ and hence

$$u(z) = A\{v_1(a)\,v_2(z) - v_2(a)\,v_1(z)\}$$

where A is a constant.

If C encloses the singularity $t = a$, then, since

$$(1 - zt/a)^{2m+1} = \{(1-z) - (t-a)\,z/a\}^{2m+1},$$

it follows that $(z-1)^{m+1}$ is a factor of $u(z)$ and hence

$$u(z) = B\{v_1(1)\,v_2(z) - v_2(1)\,v_1(z)\}$$

where B is a constant.

Hence for these contours, if the equation (2.2) is to be an integral equation the function $v(t)$ in the integrand must be the same function of t as the corresponding $u(z)$ is of z. Using the relations (1.6) we have, therefore, solutions of the integral equation.

$$v(z) = \frac{\lambda}{2\pi i} \int_C (1 - zt/a)^{2m+1} \frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}} \tag{3.1}$$

for contours encircling one singularity only as:

$$\begin{aligned} &\text{around } t = 0,\ V_0(z) = v_2(0)\,v_2(z) - v_1(0)\,v_1(z), \\ &\quad\ ,, \quad t = 1,\ V_1(z) = v_2(1)\,v_2(z) - v_1(1)\,v_1(z), \\ &\quad\ ,, \quad t = a,\ V_a(z) = v_2(a)\,v_2(z) - v_1(a)\,v_1(z). \end{aligned} \tag{3.2}$$

4. *Value of* λ.

The constant λ involves the characteristic value of h for any solution of the integral equation (3.1) and may be evaluated in terms of the coefficient c_m of z^m in $v_1(z)$. Let λ_0, λ_1, λ_a denote the values of λ for the three contours considered.

Substituting $V_0(z)$ in (3.1) and comparing coefficients of z^m, we have

$$-v_1(0)\,a^{\frac{1}{2}}c_m = \frac{\lambda_0}{2\pi i}\int_{(0+)} \frac{(2m+1)!}{m!\,(m+1)!}\left(-\frac{1}{a}\right)^m \frac{V_0(t)\,dt}{t\,(t-1)^{m+1}(t-a)^{m+1}}$$

$$= \lambda_0 \frac{(2m+1)!}{m!\,(m+1)!}\left(-\frac{1}{a}\right)^m \frac{V_0(0)}{a^{m+1}},$$

$$V_0(0) = -v_1^2(0) = -a^{m+1}v_1(0),$$

and hence
$$\lambda_0 = (-)^m \frac{\Gamma(\frac{1}{2})\,(m+1)!}{2^{2m+1}\Gamma(m+\frac{3}{2})}\,a^{m+1}c_m. \tag{4.1}$$

We have seen that $V_1(z)$ contains the factor $(z-a)^{m+1}$, and hence $V_1(t)/(t-a)^{m+1}$ is a polynomial in $(t-1)$ of which the first term is $v_2(1)\,(t-1)^m$. Substituting $V_1(t)$ in (3.1) and comparing coefficients of z^{m+1}, we have

$$v_2(1)\,c_m = \frac{\lambda_1}{2\pi i}\int_{(1+)} \frac{(2m+1)!}{m!\,(m+1)!}\left(-\frac{1}{a}\right)^{m+1} \frac{v_2(1)\,(t-1)^m + \cdots}{(t-1)^{m+1}}\,dt.$$

$$= \lambda_1 \frac{(2m+1)!}{m!\,(m+1)!}\left(-\frac{1}{a}\right)^{m+1} v_2(1).$$

Hence $\lambda_1 = -\lambda_0$.

Similarly, substituting $V_a(z)$ in (3.1) and comparing coefficients of z^{m+1}, we find that $\lambda_a = -\lambda_0$.

Therefore

$$\lambda_a = \lambda_1 = -\lambda_0 = (-)^{m+1} \frac{\Gamma(\frac{1}{2})\,(m+1)!}{2^{2m+1}\Gamma(m+\frac{3}{2})}\,a^{m+1}c_m. \tag{4.2}$$

5. *Contour enclosing two singularities.*

Since $V_1(t)$ has the factor $(t-a)^{m+1}$ and $V_a(t)$ the factor $(t-1)^{m+1}$, we have

$$A\,V_1(z) = \frac{\lambda_1}{2\pi i}\int_{(1+)} \left(1-\frac{zt}{a}\right)^{2m+1} \frac{A\,V_1(t) + B\,V_2(t)}{\{t(t-1)\,(t-a)\}^{m+1}}\,dt$$

$$B\,V_a(z) = \frac{\lambda_1}{2\pi i}\int_{(a+)} \left(1-\frac{zt}{a}\right)^{2m+1} \frac{AV_1(t) + BV_2(t)}{\{t\,(t-1)\,(t-a)\}^{m+1}}\,dt,$$

where A and B are any constants.

Hence

$$AV_1(z) + BV_a(z) = \frac{\lambda_1}{2\pi i}\int_{(1+,\,a+)}\left(1-\frac{zt}{a}\right)^{2m+1}\frac{AV_1(t)+BV_2(t)}{\{t(t-1)(t-a)\}^{m+1}}\,dt.$$

It follows that any solution of the differential equation (1.2) when h has a characteristic value satisfies the integral equation

$$v(z) = \frac{\lambda_1}{2\pi i}\int_{(1+,\,a+)}\left(1-\frac{zt}{a}\right)^{2m+1}\frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}\,dt. \quad (5.1)$$

6. *Nucleus* $(z-t)^{2m+1}$.

Substituting a/z for z in (3.1), we have

$$z^{2m+1}v(a/z) = \lambda\int_C (z-t)^{2m+1}\frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}.$$

Hence for contours around one singularity we have, using (1.6),

$$a^{m+\frac{1}{2}}v_2(z) = \frac{\lambda_0}{2\pi i}\int_{(0+)}(z-t)^{2m+1}\frac{v_1(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}$$

$$-V_a(z) = \frac{\lambda_1}{2\pi i}\int_{(1+)}(z-t)^{2m+1}\frac{V_1(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}$$

$$-a^{2m+1}V_1(z) = \frac{\lambda_a}{2\pi i}\int_{(a+)}(z-t)^{2m+1}\frac{V_a(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}.$$

Combining these results as before, we have

$$v_2(z) = a^{-m-\frac{1}{2}}\frac{\lambda_1}{2\pi i}\int_{(1+,\,a+)}(z-t)^{2m+1}\frac{v_1(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}} \quad (6.1)$$

$$v_1(z) = a^{-m-\frac{1}{2}}\frac{\lambda_1}{2\pi i}\int_{(1+,\,a+)}(z-t)^{2m+1}\frac{v_2(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}. \quad (6.2)$$

Hence the solution of the integral equation

$$v(z) = \frac{\mu}{2\pi i}\int_{(1+,\,a+)}(z-t)^{2m+1}\frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}} \quad (6.3)$$

is $v(z) = v_2(z) \pm v_1(z)$

and $$\mu = \pm(-)^{m+1}\frac{\Gamma(\frac{1}{2})(m+1)!}{2^{2m+1}\Gamma(m+\frac{3}{2})}a^{\frac{1}{2}}c_m.$$

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Congruences on a Distributive Lattice

By H. A. Thurston

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Among the many papers on the subject of lattices I have not seen any simple discussion of the congruences on a distributive lattice. It is the purpose of this note to give such a discussion for lattices with a certain finiteness. Any distributive lattice is isomorphic with a ring of sets (G. Birkhoff, *Lattice Theory*, revised edition, 1948, p. 140, corollary to Theorem 6); I take the case where the sets are finite. All finite distributive lattices are covered by this case.

Let $\mathcal{L}$ be a lattice of subsets of S and K be a subset of S (not necessarily an element of $\mathcal{L}$). Let the definition of the relation $\mathfrak{q}_K$ between elements of the lattice be that $X\mathfrak{q}_K Y$ if and only if $X \cap K = Y \cap K$. $\mathfrak{q}_K$ is clearly an equivalence (this would be true for any lattice). In fact, $\mathfrak{q}_K$ is a congruence. For if $X \mathfrak{q}_K Y$ and $Z \in L$, then $(X \cap Z) \cap K = (X \cap K) \cap Z = (Y \cap K) \cap Z = (Y \cap Z) \cap K$ and $(X \cup Z) \cap K = (X \cap K) \cup (Z \cap K) = (Y \cap K) \cup (Z \cap K) = (Y \cup Z) \cap K$. Therefore $(X \cap Z)\, \mathfrak{q}_K\, (Y \cap Z)$ and $(X \cup Z)\, \mathfrak{q}_K\, (Y \cup Z)$.

The chief theorem is the converse of this: *If $\mathcal{L}$ is a lattice of finite subsets of S, and $\mathfrak{q}$ is a congruence on $\mathcal{L}$, then there is a subset K of S such that $\mathfrak{q} = \mathfrak{q}_K$.*

The set $X\mathfrak{q}$ of all elements in the relation $\mathfrak{q}$ to X is a sub-lattice. For if $Y\mathfrak{q}\, X\, \mathfrak{q}\, Z$, then $(Y \cap Z)\, \mathfrak{q}\, (X \cap X) = X$ and $(Y \cup Z)\, \mathfrak{q}\, (X \cup X) = X$. Let X_g and X_l be the greatest and least elements of $X\mathfrak{q}$.

Let K be $\cup X - \cup (X_g - X_l)$. (Unions are over all X of $\mathcal{L}$.) First we see that $\mathfrak{q} \subset \mathfrak{q}_K$. For $X_g - X_l \subseteq \cup X - K$. Therefore $(X_g - X_l) \cap K \subseteq K - K = \emptyset$, and so $X_g \cap K = X_l \cap K$. But $X_g \cap K \supset X \cap K \supset X_l \cap K$, giving $X \cap K = X_l \cap K$. If $X\mathfrak{q} = Y\mathfrak{q}$ then $X_l = Y_l$. Therefore $Y \cap K = Y_l \cap K = X_l \cap K = X \cap K$, and so $X\, \mathfrak{q}_K\, Y$.

We have now to see that if $X\mathfrak{q}_K Y$, then $X\mathfrak{q}Y$. We take first the case $X \supset Y$; the proof is by induction on the number of elements in $X - Y$. It is clearly true when $X - Y$ has no elements. Let $X - Y$ have n elements ($n > 0$) and $a \in X - Y$ and $X \cap K = Y \cap K$. Then a is not in K and so $a \in \cup X - K = \cup (X_g - X_l)$. Therefore there are elements P and Q of $\mathcal{L}$ such that $P\, \mathfrak{q}\, Q$ and $a \in P - Q$. Then $U\, \mathfrak{q}\, V$ where $U = Y \cup P. \cup X$ and $V = Y \cup Q. \cap X$.

Then $$Y \subseteq U \cap V \subseteq U \subseteq X. \qquad (1)$$

Now a is not in $U \cap V$ and $a \in U$. The number of elements in $X - U$ and the number of elements in $(U \cap V) - Y$ are therefore less than n. But, from (1), $K \cap Y \subseteq K \cap U \cap V \subseteq K \cap U \subseteq K \cap X$. And $X \cap K = Y \cap K$. Therefore all these are equal. Therefore we have $K \cap X = K \cap U$, $U \subset X$, and the number of elements in $X - U$ is less than n. Therefore $U \mathfrak{q} X$. In the same way, $Y \mathfrak{q} U \cap V$. But $U \mathfrak{q} U \cap V$. Therefore $X \mathfrak{q} Y$.

Now let (X, Y) be any element of $\mathfrak{q}_K$. Then $X \cap K = X \cap Y \cap K$, and $X \supset X \cap Y$, and so $X \mathfrak{q} X \cap Y$. In the same way, $Y \mathfrak{q} Y \cap X$, and so $X \mathfrak{q} Y$.

Note. A similar theorem for complemented modular lattices is given by Birkhoff, *loc. cit.*, p. 119, Theorem 5.

Definition: If $\mathfrak{p}$ and $\mathfrak{r}$ are any two relations, then $\mathfrak{pr}$ is the relation for which $X \mathfrak{pr} Y$ if and only if there is a Z for which $X \mathfrak{p} Z$ and $Z \mathfrak{r} Y$.

We can now prove that

If $\mathfrak{p}$ and $\mathfrak{r}$ are any two congruences on a lattice of finite subsets, then $\mathfrak{pr} = \mathfrak{rp}$.

Let $\mathfrak{p}$ be $\mathfrak{q}_P$ and $\mathfrak{r}$ be $\mathfrak{q}_R$. If $A \mathfrak{pr} B$, then, for some element C of the lattice, $A \mathfrak{q}_P C \mathfrak{q}_R B$, and so $A \cap P = C \cap P$ and $B \cap R = C \cap R$.

Then $A \cap P \cap R = C \cap P \cap R = C \cap R \cap P = B \cap R \cap P$. Let D be $(A \cap R) \cup (B \cap P)$. Then $D \cap R = (A \cap R) \cup (B \cap P \cap R) = (A \cap R) \cup (A \cap P \cap R) = A \cap R$. Therefore $A \mathfrak{r} D$. In the same way, $D \mathfrak{p} B$. Therefore $A \mathfrak{rp} B$, and so $\mathfrak{pr} \subseteq \mathfrak{rp}$. Similarly, $\mathfrak{rp} \subseteq \mathfrak{pr}$.

Note. This theorem was proved for relatively complemented lattices by R. P. Dilworth, *Annals of Mathematics*, 50(1950), 348.

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A Determinantal Expansion for a Class of Definite Integral

Part 2.

By L. R. Shenton.

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A. *Introduction.*

In a previous paper (Shenton, 1953) we have given an expansion for integrals of the form $\int \frac{A(x)\,B(x)}{C(x)}\,w(x)\,dx$. This expansion may be expressed as a determinantal quotient or Schweinsian series. In the present paper we state more general terms under which the expansion holds and consider the case when the limits of integration are infinite and the weight function of the form $A(x)\,e^{-x}$ or $A(x)\,e^{-\frac{1}{2}x^2}$. In particular we give expansions for $\int_0^\infty \frac{e^{-ax}x^{s-1}}{C(x)}\,dx$, the Psi function, and $\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{C(x)}\,dx$, where $C(x)$ is a positive polynomial.

We take this opportunity to remark that the method in this and the previous paper is closely related to the expansion of certain definite integrals as continued fractions. Indeed Tchebycheff (1859) uses an interpolation formula to give an expansion of a function in terms of orthogonal functions, these functions appearing as the denominators of the convergents of a continued fraction. As examples he gives

$$\frac{1}{\pi}\int_{-1}^{1} \frac{du}{(x-u)\sqrt{(1-u^2)}} = \frac{1}{x}-\frac{1}{2x}-\frac{1}{2x}-\cdots,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-ku^2}\,du}{x-u} = \frac{1}{\lambda x}-\frac{1}{\lambda x}-\frac{2}{\lambda x}-\frac{3}{\lambda x}-\cdots, \lambda=\sqrt{(2k)},$$

$$\int_0^\infty \frac{e^{-ku}\,du}{x-u} = \frac{1}{kx-1}-\frac{1^2}{kx-3}-\frac{2^2}{kx-5}-\cdots.$$

The general method of expressing a definite integral as a continued fraction (*i.e.* a determinantal quotient of continuants) has been to

convert the integral into an infinite series, convergent or divergent, and to express this series as a continued fraction. With this procedure orthogonal polynomials appear in certain cases (Wall, 1945, pp. 192-202).

Romanovsky (1927) has treated Tchebycheff's method of interpolation and suggested that the interpolatory function might be used for points outside the range and for the case when the function is defined at an infinite number of points. The method we use is an extension of this and leads to a generalised type of continued fraction. Questions of convergence can be settled by an appeal to Parseval's theorem in the theory of orthogonal functions.

B. *Parseval's Theorem.*

We shall consider the formal expansion

$$\int_a^b \frac{A(x)\,B(x)\,w(x)}{C(x)}\,dx = \sum_{s=1}^{\infty} \frac{|\alpha_0, \gamma_{01}, \gamma_{12}, \ldots, \gamma_{s-1,s}| \cdot |\beta_0, \gamma_{01}, \gamma_{12}, \ldots \gamma_{s-1,s}|}{\Delta_{s-1}\,\Delta_s} \tag{1}$$

$$= -\lim_{s\to\infty} \begin{vmatrix} 0 & \alpha_0 & \alpha_1 & \cdot\cdot & \alpha_s \\ \beta_0 & \gamma_{00} & \gamma_{01} & \cdot\cdot & \gamma_{0s} \\ \beta_1 & \gamma_{10} & \gamma_{11} & \cdot\cdot & \gamma_{1s} \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \beta_s & \gamma_{s0} & \gamma_{s1} & \cdot\cdot & \gamma_{ss} \end{vmatrix}, \tag{2}$$

where

$$\begin{matrix} \alpha_s \\ \beta_s \end{matrix} = \int_a^b \theta_s(x)\,w(x) \begin{matrix} A(x) \\ B(x) \end{matrix} dx, \tag{3}$$

$$\gamma_{r,s} = \gamma_{s,r} = \int_a^b \theta_s(x)\,\theta_r(x)\,C(x)\,w(x)\,dx, \tag{4}$$

$$\Delta_s \equiv |\gamma_{00}, \gamma_{11}, \ldots, \gamma_{ss}|,$$

and $p_s(x) =$

$$\begin{vmatrix} \theta_0(x) & \theta_1(x) & \cdots & \theta_s(x) \\ \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0s} \\ \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1s} \\ \vdots & \vdots & & \vdots \\ \gamma_{s-1,0} & \gamma_{s-1,1} & \cdots & \gamma_{s-1,s} \end{vmatrix} \div \sqrt{(\Delta_{s-1}\,\Delta_s)},$$

with $\theta_s(x)$ an arbitrary polynomial of precise degree s. The set of polynomials $\{p_s(x)\}$ is an orthonormal system with respect to the

weight function $C(x)\,w(x)$. In (2) we have introduced the notation

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \equiv \frac{|a_0, b_1, c_2|}{|b_1, c_2|}$$

and so on for other orders. If $A(x)$, $B(x)$ and $C(x)$ are polynomials of degrees l, m and n respectively, then there is the formal expansion

$$\int_a^b \frac{A(x)\,B(x)\,w(x)}{C(x)}\,dx =$$

$$(-)^n \sum_{r=0}^{\infty} \frac{k_{r+n}\,\left|\sum_{\lambda=0}^{r} a_\lambda p_\lambda(x_1),\ p_{r+1}(x_2),\ \ldots p_{r+n-1}(x_n)\right| \cdot \left|\sum_{\lambda=0}^{r} b_\lambda p_\lambda(x_1), p_{r+1}(x_2), \ldots p_{r+n-1}(x_n)\right|}{k_r\,|\,p_r\,(x_1), p_{r+1}(x_2),\ \ldots p_{r+n-1}(x_n)\,| \cdot |\,p_{r+1}(x_1), p_{r+2}(x_2),\ \ldots\ p_{r+n}(x_n)\,|} \qquad (6)$$

where $\{p_s(x)\}$ is an orthonormal set with respect to $w(x)$ on $(a,\ b)$, k_s being the highest coefficient in $p_s(x)$, $C(x)$ has the roots $x_j, j=1, 2, \ldots, n$ (assumed distinct), and $A(x) = \sum_{\lambda=0}^{l} a_\lambda p_\lambda(x)$, $B(x) = \sum_{\lambda=0}^{m} b_\lambda p_\lambda(x)$.

In the expression $\sum_{\lambda=0}^{r} a_\lambda p_\lambda(x)$ it is to be understood that $a_\lambda = 0$ if $\lambda > l$, and similarly in $\sum_{\lambda=0}^{r} b_\lambda p_\lambda(x)$, $b_\lambda = 0$ if $\lambda > m$.

We now consider the expansions (1), (2) and (6) in relation to Parseval's theorem, which may be stated as follows :

P. 1. Finite Range.[1] *Let*

(i) $w(x)$ *be a non-negative and measurable weight function such that* $\int_a^b w(x)\,dx > 0$ *and* $\int_a^b x^n w(x)\,dx$ *exists for* $n = 0, 1, \ldots,$

(ii) $f(x)\sqrt{w(x)}$ *be of the class* $L^2(a, b)$,

(iii) $\{p_s(x)\}\sqrt{w(x)}$ *be an orthonormal system with* $p_s(x)$ *a polynomial in* x *of precise degree* s.

Then $$\int_a^b |f(x)|^2 w(x)\,dx = \sum_{r=0}^{\infty} |f_r|^2,$$

where $$\int_a^b f(x)\,p_r(x)\,w(x)\,dx = f_r.$$

Similarly $$\int_a^b f(x)\,g(x)\,w(x)\,dx = \sum_{r=0}^{\infty} f_r g_r$$

provided $g(x)\sqrt{w(x)}$ *also belongs to* $L^2(a, b)$.

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[1] Szegö, G., *Orthogonal Polynomials* (1939), p. 39.

P. 2. Range 0 to ∞. *The theorem holds if the conditions of* P. 1 *are satisfied with the weight function* $w(x) = e^{-x}x^a\bar{w}(x), a > -1$, *where* (*a*) $\bar{w}(x)$ *is a non-negative bounded measurable function, or* (*b*) $w(x)$ *is a non-negative polynomial of given degree.*

P. 3. Range $-\infty$ to ∞. *The theorem holds if the conditions of* P. 1 *are satisfied with* $w(x) \equiv e^{-x^2}\bar{w}(x)$, *where* $\bar{w}(x)$ *satisfies* (*a*) *or* (*b*) *of* P. 2.

The statements in P. 2 and P. 3 when $w(x) = 1$ have been given by Szegö (*loc. cit.*, pp. 104-106) who extended a method due to J. v. Neumann for a weight function of the form e^{-x} (see Courant, R. and Hilbert, D., *Methoden der Mathematischen Physik*, Vol. 1 (Berlin 1931), pp. 81-2). Following v. Neumann and Szegö, we can deduce P. 2 from P. 1 provided it can be shown that if m is a non-negative integer there exists for every $\epsilon > 0$ a polynomial $p_{n-1}(x)$ such that

$$S^2 = \int_0^\infty e^{-x}\, x^a\, \bar{w}(x) \left\{e^{-mx} - p_{n-1}(x)\right\}^2 dx < \epsilon. \tag{7}$$

P. 2 (a) follows with $\bar{w}(x) = 1$, and P. 3 (a) may be deduced from this. P. 2 (b) may be proved by an extension of the Neumann-Szegö method. We require the following properties of the Laguerre polynomials:

$$n!\, L_n^a(x) = e^x x^{-a}\left(\frac{d}{dx}\right)^n e^{-x}x^{a+n}, \qquad n = 0, 1, \ldots, \tag{8}$$

$$\int_0^\infty e^{-x}x^a L_n^a(x)\, L_m^a(x)\, dx = \frac{(n+a)!}{n!}\delta_{n,m}, \qquad n, m = 0, 1, \ldots, \tag{9}$$

$$(1-\omega)^{a+1}\sum_{r=0}^{\infty}\omega^r L_r^a(x) = \exp\{-\omega x/(1-\omega)\}, \qquad |\omega| < 1, \tag{10}$$

$$L_n^a(x) = L_n^{a+r}(x) - \binom{r}{1}L_{n-1}^{a+r}(x) + \binom{r}{2}L_{n-2}^{a+r}(x) - \ldots \quad . \tag{11}$$

Suppose now that

$$\begin{aligned} \bar{w}(x) &= a + bx + cx^2, \qquad a \neq 0 \\ &\leqq (|a| + |b| + |c|)(1 + x^2) \\ &= k(1 + x^2). \end{aligned}$$

Then

$$S^2 \leqq k\int_0^\infty e^{-x}\left(x^a + x^{a+2}\right)\left(e^{-mx} - p_{n-1}(x)\right)^2 dx.$$

Take $p_{n-1}(x) = (1-\omega)^{a+1}\sum_{s=0}^{n-1}\omega^s L_s^a(x), \qquad \omega = m/(m+1)$ (12)

so that

$$S^2 \leqq k\int_0^\infty e^{-x}x^\alpha\Big[(1-\omega)^{\alpha+1}\Sigma\,\omega^s L_s^\alpha(x)\Big]^2 dx$$
$$+k\int_0^\infty e^{-x}x^{\alpha+2}\Big[(1-\omega)^{\alpha+1}\omega^n L_{n-2}^{\alpha+2}(x)-(1-\omega)^{\alpha+1}\omega^n(2-\omega)L_{n-1}^{\alpha+2}(x)$$
$$+(1-\omega)^{\alpha+3}\Sigma\,\omega^s L_s^{\alpha+2}(x)\Big]^2 dx \quad (13)$$

after using (10) and (11), the rearrangement of terms being justified since (10) is absolutely convergent. Hence

$$S^2 \leqq k(1-\omega)^{2\alpha+2}\omega^{2n}\frac{(n+\alpha+2)!}{(n-2)!}\Big\{\frac{F(1,n+\alpha+1;n+1;\omega^2)}{n(n-1)(n+\alpha+2)(n+\alpha+1)}$$
$$+(1-\omega)^4\frac{F(1,n+\alpha+3;n+1;\omega^2)}{n(n-1)}+\frac{1}{(n+\alpha+2)(n+\alpha+1)}$$
$$+\frac{(2-\omega)^2}{(n-1)(n+\alpha+2)}\Big\}$$

with the usual notation for the hypergeometric series. Term-by-term integration is justified since[1] $|L_n^\alpha(x)| < e^x(n+\alpha)!/n!$, $x \geqq 0$, so that $\sum_{r,s}\omega^{r+s}L_r^\alpha(x)L_s^\alpha(x)$ converges uniformly for x in $(0, A)$, $A > 0$ fixed, and by Schwarz's inequality

$$\sum_{r,s=n}^\infty\int_0^\infty\omega^{r+s}e^{-x}x^\alpha|L_r^\alpha(x)|\,.\,|L_s^\alpha(x)|\,dx \leqq \sum_{r,s=n}^\infty\omega^{r+s}\Big\{\frac{(r+\alpha)!}{r!}\frac{(s+\alpha)!}{s!}\Big\}^{\frac{1}{2}}$$

converges.[2] Since $\omega < 1$ it is seen that, for $\epsilon > 0$, $n \geqq n(\epsilon, \alpha)$ exists so that $S^2 < \epsilon$. A similar proof applies to $w(x)$ of any given degree. P. 3 (b) follows from this (see Szegö, *loc. cit.*, p. 105 (3)).

C. *Illustrative Examples.*

C. 1. Let $I(p, q) = \int_{-1}^{1}\frac{dx}{(x^2+2px+q)\sqrt{(1-x^2)}}$,

where $x^2+2px+q > 0$ for $-1 \leqq x \leqq 1$.

The conditions of P. 1 are satisfied with $w(x) = (x^2+2px+q)/\sqrt{(1-x^2)}$ and $f(x) = 1/(x^2+2px+q)$. Hence using (2) with

$$A(x) = B(x) = 1, \qquad C(x) = x^2+2px+q, \qquad w(x) = 1/\sqrt{(1-x^2)},$$

we have[3] with

$$\theta_s(x) = \sqrt{(2/\pi)}\cos s\phi, \qquad \cos\phi = x, \qquad s = 1, 2, \ldots$$
$$\theta_0(x) = \sqrt{(1/\pi)}$$

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[1] Uspensky, J. V., *Ann. Math.* (2), 28, 608.

[2] Bromwich, T. J. I'A., *Introduction to the theory of infinite series* (London, 1926), p. 500, B.

[3] See Szegö, *loc. cit.*, pp. 30-32.

the values

$$\gamma_{s,s} = q + \tfrac{1}{2}, \qquad s = 0, 2, 3\ldots, \qquad \gamma_{11} = q + \tfrac{3}{4}$$

$$\gamma_{s,s+1} = p, \qquad s = 1, 2, \ldots, \qquad \gamma_{01} = p\sqrt{2}$$

$$\gamma_{s,s+2} = \tfrac{1}{4}, \qquad s = 1, 2 \ldots, \qquad \gamma_{02} = 1/2\sqrt{2}$$

$$\gamma_{s,r} = 0, \qquad r > s + 2$$

$$\alpha_s = \beta_s = 0, \qquad s \neq 0$$

$$\alpha_0 = \beta_0 = 2\sqrt{(2/\pi)}$$

so that after slight simplification

$$\frac{2}{\pi} I(p,q) = -\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \cdot\cdot \\ 1 & \tfrac{1}{2}q+\tfrac{1}{4} & p & \tfrac{1}{4} & 0 & \cdot\cdot \\ 0 & p & q+\tfrac{3}{4} & p & \tfrac{1}{4} & \cdot\cdot \\ 0 & \tfrac{1}{4} & p & q+\tfrac{1}{2} & p & \cdot\cdot \\ 0 & 0 & \tfrac{1}{4} & p & q+\tfrac{1}{2} & \cdot\cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdot\cdot \end{vmatrix}, \tag{14}$$

the expansion providing an increasing sequence.

Similarly, if $q - p^2 \neq 0$, then from

$$\int_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)}} = (q-p^2) I + \int_{-1}^{1} \frac{(x+p)^2\, dx}{(x^2+2px+q)\sqrt{(1-x^2)}}$$

we have

$$\frac{2}{\pi}(q-p^2) I(p,q) = \begin{vmatrix} 2 & p & 1 & 0 & 0 & \cdot\cdot \\ p & \tfrac{1}{2}q+\tfrac{1}{4} & p & \tfrac{1}{4} & 0 & \cdot\cdot \\ 1 & p & q+\tfrac{3}{4} & p & \tfrac{1}{4} & \cdot\cdot \\ 0 & \tfrac{1}{4} & p & q+\tfrac{1}{2} & p & \cdot\cdot \\ 0 & 0 & \tfrac{1}{4} & p & q+\tfrac{1}{2} & \cdot\cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}, \tag{15}$$

and this gives a decreasing sequence for $I(p, q)$ if $q > p^2$.

An alternative expansion follows from (6) with $x^2+2px+q \equiv (x-\cos\theta_1)(x-\cos\theta_2)$, where θ_1 and θ_2 are complex, $\theta_1 \neq \theta_2$:

$$I(p,q) = \sum_{s=1}^{\infty} \frac{2\pi(\cos s\theta_1 - \cos s\theta_2)^2}{|\cos(s-1)\theta_1, \cos s\theta_2| \cdot |\cos s\theta_1, \cos(s+1)\theta_2|}. \tag{16}$$

The expansions (14)-(16) represent simple generalisations of the continued fraction development

$$\frac{1}{\pi}\int_{-1}^{1}\frac{dx}{(z-x)\sqrt{(1-x^2)}}=\frac{1}{z-}\frac{\frac{1}{2}}{z-}\frac{\frac{1}{4}}{z-}\frac{\frac{1}{4}}{z-}\cdots \qquad |z|>1$$

$$=\sum_{s=1}\sec(s-1)\theta\sec s\theta \qquad \text{with} \qquad \cos\theta=z.$$

Similar results hold for $\int_{-1}^{1}\frac{\sqrt{(1-x^2)}\,dx}{x^2+2px+q}$, $\int_{-1}^{1}\frac{1}{x^2+2px+q}\sqrt{\frac{1-x}{1+x}}dx$ and for $C(x)$ a polynomial of higher degree than the second.

C. 2. We next consider

$$\zeta(b,a)=\int_0^\infty\frac{e^{-ax}x^{b-1}dx}{1-e^{-x}}=\Gamma(b)\sum_{s=0}^{\infty}(s+a)^{-b},\ a>0,\ b>1, \quad (17)$$

$$\text{and}\ \theta(b,a)=\int_0^\infty\frac{e^{-ax}x^{b-1}dx}{1+e^{-x}}=\Gamma(b)\sum_{s=0}^{\infty}(-)^s(s+a)^{-b},\ a>0,\ b>0. \quad (18)$$

With $\bar{w}(x)=(1-e^{-x})/x$, which is non-negative, measurable and bounded ($\leqq 1$), $w(x)=e^{-ax}x^{b-3}(1-e^{-x})$, and $f(x)=x/(1-e^{-x})$ so that $f(x)\sqrt{w(x)}$ belongs to $L^2(0,\infty)$, the conditions of P. 2. (a) are satisfied. In (2) we take $w(x)=e^{-ax}x^{b-2}$, $C(x)=(1-e^{-x})/x$, $A(x)=B(x)=1$, $\theta_r(x)=x^r$, so that $\alpha_r=\beta_r=\Gamma(r+b-1)\,a^{1-b-r}$, $\gamma_{r,s}=-\Gamma(r+s+b-2)\,\Delta a^{2-r-s-b}$, where $\Delta a^n\equiv(a+1)^n-a^n$.

Thus

$$\zeta(b,a)=\begin{vmatrix} 0 & \Gamma(b-1)a^{1-b} & \Gamma(b)a^{-b} & \cdots \\ \Gamma(b-1)a^{1-b} & \Gamma(b-2)\Delta a^{2-b} & \Gamma(b-1)\Delta a^{1-b} & \cdots \\ \Gamma(b)a^{-b} & \Gamma(b-1)\Delta a^{1-b} & \Gamma(b)\Delta a^{-b} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix}. \quad (19)$$

In the special case $b=2$, $\Gamma(b-2)\,\Delta a^{2-b}$ must be replaced by $-\log(1+1/a)$.

Similarly

$$\theta(b,a)=-\begin{vmatrix} 0 & \Gamma(b)a^{-b} & \Gamma(b+1)a^{-b-1} & \cdots \\ \Gamma(b)a^{-b} & \Gamma(b)\nabla a^{-b} & \Gamma(b+1)\nabla a^{-b-1} & \cdots \\ \Gamma(b+1)a^{-b-1} & \Gamma(b+1)\nabla a^{-b-1} & \Gamma(b+2)\nabla a^{-b-2} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} \quad \begin{matrix} a>0, \\ ,b>0, \\ (20) \end{matrix}$$

where $\nabla a^r=a^r+(a+1)^r$.

Again, using $\theta(b,a)=\Gamma(b)\,a^{-b}-\theta(b,a+1)$, we find that

$$\theta(b,a) = \begin{vmatrix} \Gamma(b)\,a^{-b} & \Gamma(b)\,(a+1)^{-b} & \Gamma(b+1)\,(a+1)^{-b-1} & \cdot & \cdot \\ \Gamma(b)\,(a+1)^{-b} & \Gamma(b)\,\nabla\,(a+1)^{-b} & \Gamma(b+1)\nabla(a+1)^{-b-1} & \cdot & \cdot \\ \Gamma(b+1)\,(a+1)^{-b-1} & \Gamma(b+1)\,\nabla\,(a+1)^{-b-1} & \Gamma(b+2)\,\nabla\,(a+1)^{-b-2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \quad \begin{matrix} a>0 \\ b>0. \end{matrix} \qquad (21)$$

The expansions (19) and (20) are positive non-decreasing sequences while (21) is a positive non-increasing sequence. As a numerical illustration we take $b=2$, $a=1$ in (20) and (21) for which $2\,\theta(2,1) = \sum_1^\infty n^{-2}$. For the first three approximations we have

(20)	(21)
$8/5 = 1{\cdot}6$	$43/26 = 1{\cdot}654$
$152/93 = 1{\cdot}634$	$8774/5332 = 1{\cdot}6455$
$33168/20187 = 1{\cdot}64304$	$11534061/7011798 = 1{\cdot}644950.$

Thus $1{\cdot}64304 < \Sigma\, n^{-2} < 1{\cdot}644950$, the correct value being $1{\cdot}644934$. Similarly from (19) we find for $\Sigma\, n^{-2}$

$$1/w = 1{\cdot}44 \qquad (w = \ln 2)$$
$$(4w-1)/(3w-1) = 1{\cdot}6421$$
$$(104w-42)/(74w-33) = 1{\cdot}64475$$
$$(16272w-7790)/(11178w-5627) = 1{\cdot}644928,$$

so that the fourth approximation is in error by 0·000,006. We note in passing that continued fractions for $\sum_{s=0}^\infty (a+s)^{-b}$ in the particular cases $b=2$ and $b=3$ have been given by Stieltjes (1890) and rediscovered, although by a different method, by Rogers (1905). For example,

$$\sum_{s=0}^\infty (a+s)^{-2} = \frac{1}{a-\frac{1}{2}} + \frac{a_1}{a-\frac{1}{2}} + \frac{a_2}{a-\frac{1}{2}} + \cdots, \qquad a_p = \frac{p^4}{4\,(4p^2-1)}.$$

With $a=1$ in this, the eleventh and twelfth convergents to Σn^{-2} are $1{\cdot}65245$ and $1{\cdot}63856$, indicating a slower rate of convergence than (19)-(21).

C. 3. *The Psi function and related integrals.*

We have

$$\Psi(t) = \ln t - \int_0^\infty e^{-xt}\left(\frac{e^x}{e^x-1} - \frac{1}{x}\right) dx, \qquad t>0.$$

Take $w(x) = e^{-tx}\bar{w}(x)$, where $\bar{w}(x) = (1-e^{-x})(x-1+e^{-x})/x^3$ is a non-negative bounded measurable function (its value for $x=0$ being taken as $\frac{1}{2}$). Put $f(x) = x/(1-e^{-x})$, so that $f(x)\sqrt{w(x)}$ belongs to $L^2(0, \infty)$. Then the conditions of P.2 (a) are satisfied. In (2), with $A(x) = B(x) = 1$,

$C(x) = (1-e^{-x})/x$, $w(x) = (x-1+e^{-x})e^{-tx}/x^2$, we have, taking $\theta_r = x^r$,

$$a_r = \beta_r = \int_0^\infty e^{-tx} x^{r-2}(x-1+e^{-x})\,dx,$$

$$\gamma_{r,s} = \gamma_{s,r} = \gamma_u = \int_0^\infty e^{-tx} x^{u-3}(x-1+e^{-x})(1-e^{-x})\,dx, \quad u = r+s = 0, 1, 2, \ldots$$

Thus
$$\begin{cases} a_0 = -1 + (1+t)\ln(1+t^{-1}) \\ a_1 = t^{-1} - \ln(1+t^{-1}) \\ a_r = \left(-\dfrac{d}{dt}\right)^r a_0. \end{cases}$$

Similarly
$$\begin{cases} \gamma_{00} = \gamma_0 = -\frac{1}{2} + \frac{1}{2}t(2+t)\ln t - (1+t)(2+t)\ln(1+t) + \frac{1}{2}(2+t)^2\ln(2+t) \\ \gamma_{01} = \gamma_{10} = \gamma_1 = -(1+t)\ln t + (3+2t)\ln(1+t) - (2+t)\ln(2+t) \\ \gamma_{02} = \gamma_{11} = \gamma_{20} = \gamma_2 = \ln t - 2\ln(1+t) + \ln(2+t) + 1/t(1+t) \\ \gamma_r = \left(-\dfrac{d}{dt}\right)^r \gamma_0. \end{cases}$$

Hence $\Psi(t) = \dfrac{d}{dt}\ln\Gamma(t)$

$$= \begin{vmatrix} \ln t & a_0 & a'_0 & a''_0 & \cdot & \cdot \\ a_0 & \gamma_0 & \gamma'_0 & \gamma''_0 & \cdot & \cdot \\ a'_0 & \gamma'_0 & \gamma''_0 & \gamma'''_0 & \cdot & \cdot \\ a''_0 & \gamma''_0 & \gamma'''_0 & \gamma^{iv}_0 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}, \quad t > 0, \tag{22}$$

in which superscripts denote derivatives. This is a non-increasing sequence. A non-decreasing sequence is found from

$$\Psi(t) = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-xt}}{1-e^{-x}}\right)dx, \qquad t > 0. \tag{23}$$

In (2) we take $A(x) = B(x) = 1$, $\quad C(x) = (1 - e^{-x})/x$;
$w(x) = (e^{-x} - e^{-2x} - xe^{-tx})/x^2$, $t \geqq 1{\cdot}5$, the restriction on t being necessary to ensure $w(x) \geqq 0$. With $\theta_r = x^r$ we find

$$a_r = \beta_r = \int_0^\infty (e^{-x} - e^{-2x} - xe^{-tx})\, x^{r-2}\, dx,$$

and in particular

$$a_0 = 1 - 2\ln 2 + \ln t, \qquad a_1 = \ln 2 - t^{-1}.$$

Similarly

$$\gamma_{r,s} = \gamma_{s,r} = \gamma_u = \int_0^\infty (e^{-x} - 2e^{-2x} + e^{-3x} - xe^{-xt} + xe^{-x(t+1)})\, x^{u-3}\, dx$$

where $\quad u = r + s = 0, 1, \ldots,$

and in particular

$$\gamma_{00} = \gamma_0 = \tfrac{1}{2} + 4\ln 2 - \frac{9}{2}\ln 3 + (1+t)\ln(1+t) - t\ln t$$

$$\gamma_{10} = \gamma_{01} = \gamma_1 = -4\ln 2 + 3\ln 3 + \ln t - \ln(1+t)$$

$$\gamma_{20} = \gamma_{11} = \gamma_{02} = 2\ln 2 - \ln 3 - 1/t(1+t).$$

As a numerical example put $t = 1$ in (22), so that

$$\Psi(1) = \begin{vmatrix} 0 & \cdot 386294 & \cdot 306853 & \cdot 500000 & . & . \\ \cdot 386294 & \cdot 284872 & \cdot 169899 & \cdot 212318 & . & . \\ .306853 & \cdot 169899 & \cdot 212318 & \cdot 416667 & . & . \\ \cdot 500000 & \cdot 212318 & \cdot 416667 & 1{\cdot}133889 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{vmatrix},$$

from which we have the first three approximations to Euler's constant $C = 0{\cdot}577216$, namely, $\cdot 52383$, $\cdot 57651$, $.57718$. Similarly, from the expansion corresponding to (23) we have

$$\Psi(2) = \begin{vmatrix} 0 & \cdot 306853 & \cdot 193147 & \cdot 250000 & . & . \\ \cdot 306853 & \cdot 238376 & \cdot 117783 & \cdot 121015 & . & . \\ \cdot 193147 & \cdot 117783 & \cdot 121015 & \cdot 194444 & . & . \\ \cdot 250000 & \cdot 121015 & \cdot 194444 & \cdot 435185 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{vmatrix},$$

so that using the recurrence relation $\Psi(1+t) = \Psi(t) + t^{-1}$ we have $C < \cdot 60500, \cdot 57754, \cdot 57723$. Hence $\cdot 57718 < C < \cdot 57723$.

A similar type of integral appears for

$$J(t) = \ln \Gamma(t) - (t - \tfrac{1}{2}) \ln t - \tfrac{1}{2} \ln 2\pi$$
$$= \int_0^\infty \left(\tfrac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-tx}}{x}\, dx, \qquad t > 0,$$

for which Stieltjes (1894) has given a continued fraction. With $w(x) = e^{-tx}\, \bar{w}(x)$, where $\bar{w}(x) = \{x - 2 + (x+2)\, e^{-x}\}\, (1 - e^{-x})/x^4$, and $f(x) = x/(1 - e^{-x})$, so that $\bar{w}(x)$ is non-negative, bounded and measurable, and $f(x)\sqrt{w(x)}$ belongs to $L^2(0, \infty)$, the conditions of P. 2 (a) are satisfied. It may be verified that

$$2J(t) = \begin{vmatrix} 0 & u^0 & u^1 & u^2 & . & . \\ u^0 & v^0 & v^1 & v^2 & . & . \\ u^1 & v^1 & v^2 & v^3 & . & . \\ u^2 & v^2 & v^3 & v^4 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{vmatrix}, \quad t > 0, \quad (24)$$

where $u^r \equiv \left(-\frac{d}{dt} \right)^r \left\{ t + \tfrac{1}{2} - t(1+t) \ln (1 + t^{-1}) \right\}$,

$$v^r \equiv \left(-\frac{d}{dt} \right)^r \left\{ -\frac{(3t^2 + 2t^3)}{6} \ln t + \frac{2(1+t)^3}{3} \ln (1+t) \right.$$
$$\left. - \frac{(2+t)^2 (1+2t)}{6} \ln (2+t) + \frac{2(1+t)}{3} \right\}$$

C. 4. *Integrals of the form* $\int_{-\infty}^{\infty} \frac{[A(x)]^2}{C(x)}\, e^{-\frac{1}{2}x^2}\, dx, \qquad C(x) > 0.$

As an illustration we consider in particular

$$G(a, b) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}\, dx}{x^2 + 2ax + b}, \qquad b > a^2. \quad (25)$$

The conditions of P. 3 (b) are satisfied with $w(x) = \frac{1}{\sqrt{(2\pi)}}\, e^{-\frac{1}{2}x^2}\, \bar{w}(x)$ where $\bar{w}(x) = x^2 + 2ax + b$, $f(x) = g(x) = 1/\bar{w}$, so that $f(x)\sqrt{w(x)}$ belongs to $L^2(-\infty, \infty)$. In (2), take $A(x) = B(x) = 1$, $C(x) = \bar{w}(x)$, $w(x) = \frac{1}{\sqrt{(2\pi)}}\, e^{-\frac{1}{2}x^2}$ and $\theta_r(x) = H_r(x) = e^{\frac{1}{2}x^2} \left(-\frac{d}{dx} \right)^r e^{-\frac{1}{2}x^2}$. Then

$$\begin{aligned} a_r = \beta_r &= 1, && r = 0 \\ &= 0, && r = 1, 2, \ldots \end{aligned}$$

$$\gamma_{r,s} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} (H_2(x) + 2a H_1(x) + b + 1) H_r(x) H_s(x) e^{-\frac{1}{2}x^2} dx$$

$$\begin{aligned} &= (b + 2r + 1)\, r!, && r = s \\ &= 2ar! && r = s + 1 \\ &= r! && r = s + 2 \\ &= 0 && r > s + 2 \end{aligned}$$

and so

$$G(a,b) = -\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & . & . \\ 1 & b+1 & (2a)\,1! & 2! & 0 & . & . \\ 0 & (2a)\,1! & (b+3)\,1! & (2a)\,2! & 3! & . & . \\ 0 & 2! & (2a)\,2! & (b+5)\,2! & (2a)3! & . & . \\ 0 & 0 & 3! & (2a)\,3! & (b+7)3! & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix}, \tag{26}$$

which gives a non-decreasing sequence. Similarly, using

$$(b - a^2)\, G(a, b) = 1 - \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{(x+a)^2 e^{-\frac{1}{2}x^2} dx}{x^2 + 2ax + b} \tag{27}$$

we derive the non-increasing sequence

$$(b-a^2)\, G(a,b) = \begin{vmatrix} 1 & a & 1 & 0 & 0 & . & . \\ a & b+1 & (2a)\,1! & 2! & 0 & . & . \\ 1 & (2a)1! & (b+3)1! & (2a)\,2! & 3! & . & . \\ 0 & 2! & (2a)\,2! & (b+5)\,2! & (2a)\,3! & . & . \\ 0 & 0 & 3! & (2a)\,3! & (b+7)\,3! & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix}. \tag{28}$$

The series expansions derived from (6) corresponding to (26) and (28) are respectively

$$G(a,b) = \sum_{s=0}^{\infty} \frac{s!\,\{H_{s+1}(a) - H_{s+1}(\beta)\}^2}{|H_s(a), H_{s+1}(\beta)| \cdot |H_{s+1}(a), H_{s+2}(\beta)|} = \sum_{s=0}^{\infty} \frac{s!\,E_s^2}{F_s F_{s+1}} \tag{29}$$

and

$$(b-a^2)\, G(a,b) = \frac{1}{b-a^2} - \frac{1}{b-a^2} \sum_{s=0}^{\infty} \frac{s! \begin{vmatrix} a + H_1(a) & H_{s+1}(a) \\ a + H_1(\beta) & H_{s+1}(\beta) \end{vmatrix}^2}{|H_s(a), H_{s+1}(\beta)| \cdot |H_{s+1}(a), H_{s+2}(\beta)|}$$

$$= \frac{1}{b-a^2} - \frac{1}{b-a^2} \sum_{s=0}^{\infty} \frac{s!\,G_s^2}{F_s F_{s+1}}, \tag{30}$$

where α, β are the roots of $x^2 + 2ax + b = 0$. The values of E_s, F_s and G_s are readily calculated from the recurrence relation $H_{s+1}(x) = x H_s(x) - s H_{s-1}(x)$. Since

$$G_s^2 + (b - a^2) E_s^2 = -4 (b - a^2) H_{s+1}(\alpha) H_{s+1}(\beta), \qquad (31)$$

it will be seen that the difference between the $(s+1)^{th}$ approximations arising from (29) and (30) is $s! F_0/(b - a^2) F_s$. This may be used to assess the rate of convergence and also as a computational check.

It is interesting to observe that, when $a = 0$, (26) and (28) reduce to simple continuant quotients and give the even and odd part of the continued fraction

$$G(0, b) = \frac{1}{b} + \frac{1}{1} + \frac{2}{b} + \frac{3}{1} + \frac{4}{b} + \cdots, \qquad b > 0.$$

By an equivalence transformation we have the Laplace (1805) continued fraction for the incomplete normal integral, namely

$$G(0, t^2) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2} dx}{x^2 + t^2} = t^{-1} e^{\frac{1}{2}t^2} \int_t^{\infty} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{t^{-1}}{t} + \frac{1}{t} + \frac{2}{t} + \frac{3}{t} + \frac{4}{t} + \cdots, \qquad t > 0.$$

s	TERM OF (29)	Σ	TERM OF (30)	Σ
0	·500000	·5000	1·166667	1·1667
1	·071429	·5714	·214286	·9524
2	·075630	·6471	.070028	·8824
3	.042596	·6897	.035062	·8473
4	·015594	·7052	·034370	·8129
5	·023337	·7286	·007269	·8057
6	·003307	.7319	·017955	·7877
7	·012806	·7447	·001422	·7863
8	·000568	.7453	·009729	·7765
9	·007125	·7524	·000181	·7764
10	·000038	.7524	·005401	·7710
11	.004019	·7564	·000000	·7710
12	.000010	·7565	.003051	·7679
13	·002293	·7588	·000036	·7679
14	·000063	·7588	·001743	·7661
15	·001318	·7601	·000087	·7660
16	·000104	·7602	·001004	·7650
17	·000762	·7610	·000115	·7649
18	·000120	·7611	·000580	·7643
19	·000441	·7616	·000122	·7642
20	·000119	.7617	·000335	.7639

As a numerical illustration we take $2a = b = 1$, and by comparison with the incomplete normal integral continued fraction development (Burgess, 1895) we expect rather slow convergence. Evaluating $s!\, F_0/(b - a^2)\, F_s$ for $s = 20$ we find that it is approximately 0·0027, so that we have only two-figure accuracy. In the table we give the terms and partial sums for the series (29) and (30), the identity (31) being used as a check.

We conclude then that $\cdot 7617 < G(0{\cdot}5, 1{\cdot}0) < \cdot 7639$, the correct value being ·7628, 2634. The oscillatory nature of the terms is noteworthy, and this would be an awkward feature if we could not construct an enveloping sequence.

We intend to discuss later various forms for the numerators and denominators of the expansions considered here, including recurrence relations, noting the relation to the theory of continued fractions.

I am greatly indebted to the referees for a number of useful comments and criticisms, and to Dr W. Ledermann for some criticisms of Part 1.

REFERENCES.

Burgess, J. (1895). "The definite integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt$ with extended tables of values," *Trans. Roy. Soc. Edinburgh*, XXXIX, 257-321.

Laplace, P. S. (1805). *Traité de Mécanique Céleste*, Vol. 4 (Paris).

Rogers, L. J. (1905). "On the representation of certain asymptotic series as convergent continued fractions." *Proc. London Math. Soc.* (2), 4, 72-89 and 393-395.

Romanovsky, V. (1927). "Note on orthogonalising series of functions and interpolation," *Biometrika*, XIX, 93-99.

Shenton, L. R. (1953). A determinantal expansion for a class of definite integral Part 1, *Proc. Edinburgh Math. Soc.* (2), 9 (1953), 44-52.

Stieltjes, T. J. (1890). *Sur quelques intégrales définies et leur développement en fractions continues.*

(1894) *Recherches sur les fractions continues.* Oeuvres complètes (Groningen), p. 387 and pp. 520, 521.

Tchebycheff, P. L. (1859). *Sur le développement des fonctions à une seule variable.* Oeuvres, Tome 1, pp. 501-508. (St Pétersbourg, 1899).

Wall, H. S. (1945). "Note on the expansion of a power series into a continued fraction," *Bull. American Math. Soc.*, 51, 97-105.

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MANCHESTER, 1.

Diffraction by a Half-Plane

By Ll. G. Chambers

(*Received 29th March* 1950. *Read 5th May* 1950.)

1. *Introduction.*

The diffraction of a simple harmonic wave train by a straight-edged semi-infinite screen was originally discussed by Sommerfeld in 1895. The analysis is of a recondite character, involving the use of multivalued functions and Riemann surfaces (1). An alternative formulation of the problem is as an inhomogeneous Wiener-Hop integral equation, the solution of which also involves considerable difficulties (2). It is the purpose of this note to show that following Friedlander (3) it is possible by the use of parabolic co-ordinates to solve the problem by elementary methods. The method can be applied either to the case of sound or that of electromagnetism, the results being formally identical.

2. *Preliminary Analysis.*

We assume that the plane is defined by $x = 0, y > 0, -\infty < z < \infty$, the problem being two dimensional.

Let us now take parabolic co-ordinates.

$$x = l(\xi^2 - \eta^2), \qquad y = 2l\xi\eta,$$

where l is any convenient quantity of the dimensions of a length.

The ranges over which ξ, η run are $-\infty < \xi < \infty$, $0 \leqq \eta < \infty$.

By restricting η to positive values, it is possible to avoid the troubles of a two-valued function. The range of negative values of η corresponds to the second sheet of the Riemann surface corresponding to the plane cut along the positive real axis.

We write $$\square_0^2 g \equiv \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2}.$$

It can be shown that

$$\chi = \chi(\xi, \eta) = \int_0^P F(ct + l\overline{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha - l\zeta^2)\,d\zeta,$$

where $$P = \{\xi^2 + \eta^2 + \overline{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha\}^{\frac{1}{2}}$$

is a solution of $\Box_0^2 \chi = 0$, as is $\chi(\xi, -\eta)$ also. This may be done by direct differentiation or by using the transformation

$$x' = -x \sin\alpha + y \cos\alpha, \qquad y' = x \cos\alpha + y \sin\alpha,$$

when χ takes the form

$$\chi = \int_0^{P'} F(ct + 2l\xi'\eta' - l\zeta^2)\, d\zeta,$$

where $P' = \xi' + \eta'$, and then differentiating (3), $\Box_0^2$ being invariant with respect to the rotation of axes in the xy plane.

3. *Boundary Conditions.*

Under the transformation used above, the half-plane $y = 0, x > 0$ goes into $\eta = 0$.

Let Ψ be the quantity that is under consideration. For an electromagnetic case we have

$$\Psi = E_z \qquad \text{or} \qquad \Psi = H_z.$$

In the first case the boundary condition on the half-plane is $\partial\Psi/\partial y = 0$, and in the second case $\Psi = 0$, the half-plane being a perfect conductor in both cases. If we have a sound field and Ψ is the excess pressure, then $\partial\Psi/\partial y = 0$ corresponds to the case of a perfectly rigid body occupying the half-plane and $\Psi = 0$ would correspond to a free surface in the half-plane (were it realisable).

The result of applying the transformation is to make the boundary $\eta = 0$ and the boundary conditions

$$\frac{\partial\Psi}{\partial\eta} = 0 \qquad \text{or} \qquad \Psi = 0 \qquad \text{respectively.}$$

The other boundary condition that applies is that when $x \to -\infty$, *i.e.* $\eta \to \infty$, then $\Psi \sim \Psi_0$, where Ψ_0 is the incident disturbance.

4. *Satisfaction of the Boundary Conditions.*

For simplicity, we will only consider the case where $\Psi = 0$ on $\eta = 0$.

The incident field is of some arbitrary form Ψ_0 travelling in the direction $\alpha + \pi$.

Let us consider the function

$$\Psi = \tfrac{1}{2}\Psi_0(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha) - \tfrac{1}{2}\Psi_0(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha - 2l\xi\eta\sin\alpha)$$
$$+ \int_0^P F(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha - l\zeta^2)\, d\zeta$$
$$- \int_0^Q F(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha - 2l\xi\eta\sin\alpha - l\zeta^2)\, d\zeta,$$

$$P = \{\xi^2 + \eta^2 + \xi^2 - \eta^2 \cos\alpha + 2\xi\eta \sin\alpha\}^{\frac{1}{2}} \to \infty \text{ as } \eta \to \infty,$$

$$Q = \{\xi^2 + \eta^2 + \xi^2 - \eta^2 \cos\alpha - 2\xi\eta \sin\alpha\}^{\frac{1}{2}} \to -\infty \text{ as } \eta \to \infty.$$

It is obvious by inspection that

$$\Psi \text{ satisfies } \Box_0^2 \Psi = 0, \qquad \text{with } \Psi = 0 \text{ on } \eta = 0.$$

In order to satisfy the condition $\Psi \sim \Psi_0$ as $\eta \to \infty$ we must have

$$\Psi_0(ct + l\xi^2 - \eta^2 \cos\alpha + 2l\xi\eta \sin\alpha)$$
$$= \tfrac{1}{2}\Psi_0(ct + l\xi^2 - \eta^2 \cos\alpha + 2l\xi\eta \sin\alpha) - \tfrac{1}{2}\Psi_0(ct + l\xi^2 - \eta^2 \cos\alpha - 2l\xi\eta \sin\alpha)$$
$$+ \int_0^\infty F(ct + l\xi^2 - \eta^2 \cos\alpha + 2l\xi\eta \sin\alpha - l\zeta^2)\, d\zeta$$
$$- \int_0^\infty F(ct + l\xi^2 - \eta^2 \cos\alpha - 2l\xi\eta \sin\alpha - l\zeta^2)\, d\zeta$$

for $\eta \to \infty$.

The functions Ψ_0 and F must therefore be related by

$$\tfrac{1}{2}\Psi_0(\mu) = \int_0^\infty F(\mu - l\zeta^2)\, d\zeta + C. \tag{A}$$

As explained in Friedlander's paper (3) the arbitrary constant can be ignored, and the solution of the integral equation is given by

$$F(u) = \frac{1}{\pi} l^{\frac{1}{2}} \int_{-\infty}^{u} \frac{\Psi_0(z)\, dz}{(u-z)^{\frac{1}{2}}} = \frac{2l^{\frac{1}{2}}}{\pi} \int_0^\infty \Psi_0(u - t^2)\, dt \tag{B}$$

provided that

$$F(-\infty) = 0. \tag{C}$$

It is clear by inspection of (A) that if we have a continuous wave of circular frequency $\omega = kc$, $\Psi_0 = \exp(ik\mu)$, then the solution is

$$F(u) = \sqrt{2} \exp(i\pi/4) \exp(ik\mu),$$

which in fact satisfies (B) even though (C) is not true.

If we take $\Psi_0(\mu) = \exp(ik\mu)$, our incident wave is

$$\exp\{ik(ct + l\xi^2 - \eta^2 \cos\alpha + 2l\zeta\eta \sin\alpha)\},$$

which is a plane wave travelling in the direction $\alpha + \pi$,

$$\Psi = \tfrac{1}{2}\exp\{ik\rho\cos(\phi - \alpha)\} - \tfrac{1}{2}\exp\{ik\rho\cos(\phi + \alpha)\}$$
$$+ \exp\{ik\rho\cos(\phi - \alpha)\}\sqrt{2}\exp(i\pi/4)\int_0^P \exp(-ikl\zeta^2)\, d\zeta$$
$$- \exp\{ik\rho\cos(\phi + \alpha)\}\sqrt{2}\exp(i\pi/4)\int_0^Q \exp(-ikl\zeta^2)\, d\zeta,$$

the time factor having been dropped.

Up till now l has been an arbitrary length. It will be convenient to take it as the wavelength ($kl = k\lambda = 2\pi$),

$$\begin{aligned} P &= \{\xi^2 - \eta^2 + \sqrt{\xi^2 - \eta^2}\cos\alpha + 2\xi\eta\sin\alpha\}^{\frac{1}{2}} \\ &= (\rho + x\cos\alpha + y\sin\alpha)^{\frac{1}{2}}/\lambda^{\frac{1}{2}} \\ &= (\rho/\lambda)^{\frac{1}{2}}[1 + \cos(\phi - \alpha)]^{\frac{1}{2}} = (2\rho/\lambda)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi - \alpha). \end{aligned}$$

In a similar manner we find that

$$Q = (2\rho/\lambda)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi + \alpha).$$

By a little manipulation we get

$$\begin{aligned} \Psi = {} & \tfrac{1}{2}\exp\{ik\rho\cos(\phi - \alpha)\} - \tfrac{1}{2}\exp\{ik\rho\cos(\phi + \alpha)\} \\ & + \exp\{ik\rho\cos(\phi - \sigma)\}\frac{1}{\sqrt{\pi}}\exp(i\pi/4)\int_0^p \exp(-i\gamma^2)\,d\gamma \\ & - \exp\{ik\rho\cos(\phi + \alpha)\}\frac{1}{\sqrt{\pi}}\exp(i\pi/4)\int_0^q \exp(-i\gamma^2)\,d\gamma \end{aligned}$$

where $p = \sqrt{2k\rho}\cos\tfrac{1}{2}(\phi - \alpha)$, $q = \sqrt{2k\rho}\cos\tfrac{1}{2}(\phi + \alpha)$.

By making use of the fact that

$$1 = \frac{2}{\sqrt{-(i\pi)}}\int_{-\infty}^0 \exp(-i\gamma^2)\,d\gamma$$

we have immediately

$$\begin{aligned} \beta\Psi = {} & \exp\{ik\rho\cos(\phi - \alpha)\}\int_{-\infty}^p \exp(-i\gamma^2)\,d\gamma \\ & - \exp\{ik\rho\cos(\phi + \alpha)\}\int_{-\infty}^q \exp(-i\gamma^2)\,d\gamma \end{aligned}$$

where $\beta = \sqrt{\pi}\exp(-i\pi/4)$.

It is fairly obvious that for the boundary condition $\partial\Psi/\partial\eta = 0$ on $\eta = 0$ we carry through an almost identical analysis beginning with the quantity

$$\begin{aligned} \Psi = {} & \tfrac{1}{2}\Psi_0(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha) + \tfrac{1}{2}\Psi_0(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha \\ & - 2l\xi\eta\sin\alpha) \\ & + \int_0^P F(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha - l\zeta^2)\,d\zeta \\ & + \int_0^Q F(ct + l\sqrt{\xi^2 - \eta^2}\cos\alpha - 2l\xi\eta\sin\alpha - l\zeta^2)\,d\zeta. \end{aligned}$$

We arrive at the result

$$\begin{aligned} \beta\Psi = {} & \exp\{ik\rho\cos(\phi - \alpha)\}\int_{-\infty}^p \exp(-i\gamma^2)\,d\gamma \\ & + \exp\{ik\rho\cos(\phi + \alpha)\}\int_{-\infty}^q \exp(-i\gamma^2)\,d\gamma, \end{aligned}$$

P, Q, p, q, β having the same meanings as before.

5. *Expressions for other field components.*

The results of the foregoing analysis may be used to give formulæ for the field components which are not parallel to the edge of the half-plane. The two cases of electric vector perpendicular and parallel to the edge will be considered. In order to write down formulæ for these components, it is first necessary to write down the results of certain differentiations. Since

$$\frac{\partial F}{\partial x} = \cos\phi \frac{\partial F}{\partial \rho} - \sin\phi \frac{1}{\rho}\frac{\partial F}{\partial \phi}, \qquad \frac{\partial F}{\partial y} = \sin\phi \frac{\partial F}{\partial \rho} + \cos\phi \frac{1}{\rho}\frac{\partial F}{\partial \phi},$$

the values of $\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ can be written down fairly easily, where

$$G = \exp\{ik\rho\cos(\phi+\chi)\}\int_{-\infty}^{m} \exp(-i\gamma^2)\,d\gamma$$

and $$m = \sqrt{2k\rho}\cos\tfrac{1}{2}(\phi+\chi).$$

These values are given by

$$\frac{1}{ik}\frac{\partial G}{\partial x} = G\cos\chi + \sqrt{1/2k\rho}\exp\{-i[k\rho+\tfrac{1}{2}\pi]\}\cos\tfrac{1}{2}(\phi-\chi)$$

and

$$-\frac{1}{ik}\frac{\partial G}{\partial y} = G\sin\chi + \sqrt{1/2k\rho}\exp\{-i[k\rho-\tfrac{1}{2}\pi]\}\sin\tfrac{1}{2}(\phi-\chi).$$

We again write $$\beta = \sqrt{\pi}\exp(-i\pi/4).$$

We write also

$$G_+ = \exp\{ik\rho\cos(\phi+\alpha)\}\int_{-\infty}^{q} \exp(-i\gamma^2)\,d\gamma$$

$$G_- = \exp\{ik\rho\cos(\phi-\alpha)\}\int_{-\infty}^{p} \exp(-i\gamma^2)\,d\gamma$$

$$p = \sqrt{2k\rho}\cos\tfrac{1}{2}(\phi-\alpha), \qquad q = \sqrt{2k\rho}\cos\tfrac{1}{2}(\phi+\alpha).$$

(A) *Electric vector perpendicular to edge.*

The components which are of interest in this case are related by

$$E_x = Z_0\frac{1}{ik}\frac{\partial H_z}{\partial y}, \qquad E_y = -Z_0\frac{1}{ik}\frac{\partial H_z}{\partial x}.$$

The incident field is defined by

$$Z_0 H_z = E_0\exp\{ik(x\cos\alpha + y\sin\alpha)\}$$
$$= E_0\exp\{ik\rho\cos(\phi-\alpha)\}$$

$$E_x = E_0 \exp\{ik\rho\cos(\phi - \alpha)\}\sin\alpha$$
$$E_y = -E_0 \exp\{ik\rho\cos(\phi - \alpha)\}\cos\alpha.$$

The diffracted field is defined by

$$\beta Z_0 H_z = E_0 (G_- + G_+),$$

and performing the necessary differentiations we deduce that

$$\beta E_x/E_0 = \sin\alpha\,(G_- - G_+) - \sqrt{2/k\rho}\,\sin\tfrac{1}{2}\phi\cos\tfrac{1}{2}\alpha\,\exp\{-i(k\rho - \tfrac{1}{2}\pi)\}$$
$$\beta E_y/E_0 = -\cos\alpha\,(G_- + G_+) - \sqrt{2/k\rho}\,\exp\{-i(k\rho + \tfrac{1}{2}\pi)\}\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\alpha.$$

(B) *Electric vector parallel to edge.*

The components which are of interest in this case are related by

$$H_x = -Z_0^{-1}\frac{1}{ik}\frac{\partial E_z}{\partial y}, \qquad H_y = Z_0^{-1}\frac{1}{ik}\frac{\partial E_z}{\partial x}.$$

The incident field is defined by

$$E_z = E_0 \exp\{ik(x\cos\alpha + y\sin\alpha)\} = E_0 \exp\{ik\rho\cos(\phi - \alpha)\}$$
$$Z_0 H_x = -E_0 \exp\{ik\rho\cos(\phi - \alpha)\}\sin\alpha$$
$$Z_0 H_y = E_0 \exp\{ik\rho\cos(\phi - \alpha)\}\cos\alpha.$$

The diffracted field is defined by

$$\beta E_z = E_0 (G_- - G_+),$$

and performing the necessary differentiations we deduce that

$$\beta Z_0 H_x/E_0 = \sin\alpha\,(G_- + G_+) + \sqrt{2/k\rho}\,\exp\{-i(k\rho - \tfrac{1}{2}\pi)\}\cos\tfrac{1}{2}\phi\sin\tfrac{1}{2}\alpha$$
$$\beta Z_0 H_y/E_0 = \cos\alpha\,(G_- - G_+) - \sqrt{2/k\rho}\,\exp\{-i(k\rho + \tfrac{1}{2}\pi)\}\sin\tfrac{1}{2}\phi\sin\tfrac{1}{2}\alpha.$$

(C) *Fresnel approximation.*

The Fresnel approximation, being scalar, is applicable to any of the six components of the electromagnetic field.

If $\Psi_0 = \exp\{ik(x\cos\alpha + y\sin\alpha)\}$ defines any incident field

component in the absence of a screen, then, to the first order, the field in $y < 0$ is given by

$$\beta\Psi = \Psi_0 \int_{-\infty}^{S} \exp(-i\gamma^2)\, d\gamma$$

where $S = \sqrt{2k\rho}\,\tfrac{1}{2}\chi$ and $\chi = \pi - (\phi - \alpha)$, ρ being assumed large and χ being assumed small.

Now if χ is small

$$\tfrac{1}{2}\chi \sim \sin \tfrac{1}{2}\chi = \cos \tfrac{1}{2}(\pi - \chi) = \cos \tfrac{1}{2}(\phi - \alpha)$$

and so $$S \sim \sqrt{2k\rho}\, \cos \tfrac{1}{2}(\phi - \alpha).$$

Thus the Fresnel approximation gives the approximate result that $\beta\Psi = G_-$.

On examining the results obtained from the exact theory, we observe that $G_-\beta^{-1}$ is common to all the expressions for the six field components.

6. *Discussion.*

It is clear from the form of the result that, provided the integral equation (A) is soluble, the method of solution will hold for any shape to the incident pulse, as has been indicated by Friedlander.

Furthermore, it is possible, formally at any rate, to obtain the diffracted field from any incident field; for if we have an incident field

$$\Psi_0(x, y) = \int_C A(\alpha) \exp\{ik(x \cos \alpha + y \sin \alpha)\}\, d\alpha$$

then the resultant field will be obtained by taking the diffracted field for a plane wave, multiplying by A (α) and integrating with respect to α over the proper contour C in the complex α-plane.

The use of parabolic co-ordinates in this problem of the half-plane obviates the use of the two-valued function of Sommerfeld, and the line $\eta = 0$ corresponds to the slit in the Riemann surface at which the two sheets coalesce, negative η corresponding to the unused sheet.

Another formulation involves the replacing of the plane (for an electromagnetic field) by a sheet of magnetic or electric current and writing down the condition that, for a certain component, the sum of the incident field and the field due to the sheet vanishes along the sheet, thereby giving an integral equation of the Wiener-Hopf type for the current distribution (2). Such an integral equation is soluble only with considerable difficulty.

Examination of the conditions $p \gtrless 0$, $q \gtrless 0$ gives us further insight into the physical conditions involved. It is clear that

$$\frac{1}{\beta}\int_{-\infty}^{T} \exp(-i\gamma^2)\, d\gamma \to \begin{matrix} 1 & (T \to +\infty) \\ 0 & (E \to -\infty). \end{matrix}$$

If we now consider as an example the case where $\Psi = 0$ on $\eta = 0$ (the other case is very similar), we have three separate cases:

(1) $p > 0$, $q > 0$. At large distances from the origin the field is effectively

$$\exp\{ik\rho \cos(\phi - \alpha)\} - \exp\{ik\rho \cos(\phi + \alpha)\}$$

which is the same as that for reflection by a complete plane.

(2) $p > 0$, $q < 0$. At large distances from the origin the field is effectively

$$\exp\{ik\rho \cos(\phi - \alpha)\}$$

which is the same as the incident field.

(3) $p < 0$, $q < 0$. This is the geometrical shadow region and at large distances from the origin the field is effectively zero, the region being screened by the half plane.

7. *Acknowledgment.*

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REFERENCES.

(1) See for example, *Handbuch der Physik*, Vol. XX, 263-291, "Licht als Wellenbewegung."

(2) Copson, 1946. "An integral equation method of solving plane diffraction problems," *Proc. Roy. Soc.*, A 186, 100-118.

(3) Friedlander, 1946. "The diffraction of sound pulses. I. Diffraction by a semi-infinite plane," *Proc. Roy. Soc.*, A 186, 322-344.

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A Note about Functions in Lip α

By N. du Plessis

(*Received 29th July* 1949. *Read 4th November* 1949.)

This note gives a proof of the result:

A necessary and sufficient condition that a trigonometrical series $T(x)$ *be the Fourier series of a function* $f(x) \in \text{Lip}\,\alpha$ $(0 < \alpha < 1)$ *is that*

$$\sigma_n - \sigma_m = O(n^{-\alpha}) \text{ uniformly in } [0, 2\pi] \text{ for all } m > n,$$

where σ_n *is the* n^{th} $(C, 1)$ *mean of* $T(x)$.

Sufficiency. We have $\sigma_n - \sigma_m \to 0$ uniformly as $m, n \to \infty$ and thus there is an f such that $\sigma_n \to f$ uniformly as $n \to \infty$. Furthermore, since $\sigma_n - \sigma_m = O(n^{-\alpha})$ uniformly $(m > n)$, we have, on letting $m \to \infty$, $\sigma_n - f = O(n^{-\alpha})$ uniformly and so $f \in \text{Lip}\,\alpha$. This last follows from a theorem by de la Vallée Poussin which states that if there is a sequence of trigonometric polynomials $\{T_n(x)\}$ such that $f(x) - T_n(x) = O(n^{-\alpha})$ uniformly in $[0, 2\pi]$ then[1] $f \in \text{Lip}\,\alpha$. Since $\sigma_n \to f$ uniformly,

$$\int_0^{2\pi} f \begin{matrix}\cos\\ \sin\end{matrix} kx\,dx = \lim_{n\to\infty} \int_0^{2\pi} \sigma_n \begin{matrix}\cos\\ \sin\end{matrix} kx\,dx,$$

and it follows that the Fourier coefficients of f are the coefficients n $T(x)$.

Necessity. We have $f \in \text{Lip}\,\alpha$ and must show that $\sigma_n - \sigma_m = O(n^{-\alpha})$ for all $m \geqq n$ where $\{\sigma_n\}$ are the $(C, 1)$ means of the Fourier series of f. Now $\sigma_n - f = O(n^{-\alpha})$, by a theorem due to S. Bernstein[2] and so

$$|\sigma_n - \sigma_m| \leqq |\sigma_n - f| + |\sigma_m - f| = O(n^{-\alpha}) + O(m^{-\alpha}) = O(n^{-\alpha}).$$

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[1] C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions* (Paris, 1919), §41.

[2] Zygmund, *Trigonometrical Series* (1935), p. 62, No. 7.

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$$\sigma_n - \sigma_m = O(n^{-\alpha}) \text{ uniformly in } [0, 2\pi] \text{ for all } m > n,$$

where σ_n *is the* n^{th} $(C, 1)$ *mean of* $T(x)$.

Sufficiency. We have $\sigma_n - \sigma_m \to 0$ uniformly as $m, n \to \infty$ and thus there is an f such that $\sigma_n \to f$ uniformly as $n \to \infty$. Furthermore, since $\sigma_n - \sigma_m = O(n^{-\alpha})$ uniformly $(m > n)$, we have, on letting $m \to \infty$, $\sigma_n - f = O(n^{-\alpha})$ uniformly and so $f \in \text{Lip}\,\alpha$. This last follows from a theorem by de la Vallée Poussin which states that if there is a sequence of trigonometric polynomials $\{T_n(x)\}$ such that $f(x) - T_n(x) = O(n^{-\alpha})$ uniformly in $[0, 2\pi]$ then[1] $f \in \text{Lip}\,\alpha$. Since $\sigma_n \to f$ uniformly,

$$\int_0^{2\pi} f \begin{matrix}\cos\\ \sin\end{matrix} kx\,dx = \lim_{n\to\infty} \int_0^{2\pi} \sigma_n \begin{matrix}\cos\\ \sin\end{matrix} kx\,dx,$$

and it follows that the Fourier coefficients of f are the coefficients n $T(x)$.

Necessity. We have $f \in \text{Lip}\,\alpha$ and must show that $\sigma_n - \sigma_m = O(n^{-\alpha})$ for all $m \geqq n$ where $\{\sigma_n\}$ are the $(C, 1)$ means of the Fourier series of f. Now $\sigma_n - f = O(n^{-\alpha})$, by a theorem due to S. Bernstein[2] and so

$$|\sigma_n - \sigma_m| \leqq |\sigma_n - f| + |\sigma_m - f| = O(n^{-\alpha}) + O(m^{-\alpha}) = O(n^{-\alpha}).$$

[1] C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions* (Paris, 1919), § 41.

[2] Zygmund, *Trigonometrical Series* (1935), p. 62, No. 7.

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